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Topology of Fibre bundles and Global Aspects of Gauge Theories

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Abstract

In these lecture notes we will try to give an introduction to the use of the mathematics of fibre bundles in the understanding of some global aspects of gauge theories, such as monopoles and instantons. They are primarily aimed at beginning PhD students. First, we will briefly review the concept of a fibre bundle and define the notion of a connection and its curvature on a principal bundle. Then we will introduce some ideas from (algebraic and differential) topology such as homotopy, topological degree and characteristic classes. Finally, we will apply these notions to the bundle setup corresponding to monopoles and instantons. We will end with some remarks on index theorems and their applications and some hints towards a bigger picture.

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Contents

1 Fibre Bundles

A fibre bundle is a manifold[3] that looks locally like a product of two manifolds, but isn't necessarily a product globally. Because of their importance in modern theoretical physics, many introductory expositions of fibre bundles for physicists exist. We give a far from exhaustive list in the references. This section is mainly inspired by [1] and [2]. To get some intuition for the bundle concept, let us start off with the easiest possible example.

1.1 Invitation: the Möbius strip

Consider a rectangular strip. This can of course be seen as the product of two line segments. If now one wants to join two opposite edges of the strip to turn one of the line segments into a circle, there are two ways to go about this. The first possibility is to join the two edges in a straightforward way to form a cylinder C, as on the left hand side of figure 1. It should be intuitively clear that the cylinder is not only locally a product, but also globally; namely $C = S^1 \times L$, where L is a line segment. A fibre bundle will be called trivial if it can be described as a global product. The cylinder is trivial in this sense, because it is not very difficult to find a global diffeomorphism from $S^1 \times L$ to C.

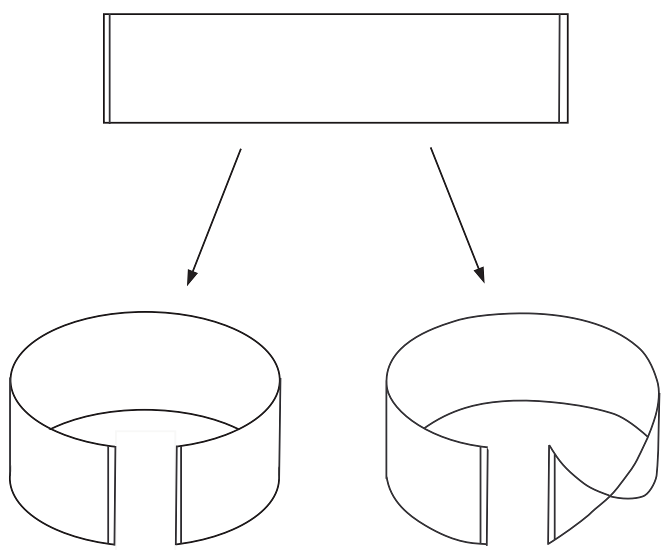

Figure 1: The cylinder and the Möbius strip.

The second way to join the edges of the strip is of course the more interesting one. Before gluing the edges together, perform a twist on one of them to arrive at a Möbius strip Mo, as shown on the right hand side of figure 1. Locally, along each open subset U of the S^1, the Möbius strip still looks like a product, $Mo = U \times L$. Globally, however, there is no unambiguous and continuous way to write a point m of Mo as a cartesian pair $(s, t) \in S^1 \times L$. The Möbius strip is therefore an example of a manifold that is not a global product, that is, of a non-trivial fibre bundle.

[3]More generally, one can define a fibre bundle as being any topological space. We will use the term fibre bundle in the more restrictive sense of manifolds throughout these notes. Whenever we will say manifold, we will mean differentiable manifold.

Since Mo is still locally a product, we can try to use this 'local triviality' to our advantage to find a useful way to describe it. Although we cannot write Mo as $S^1 \times L$, we can still project down any point m of Mo onto the circle, i.e. there is a projection π:

$$\pi : Mo \to S^1 \,, \tag{1}$$

so that, for every $x \in S^1$, its inverse image is isomorphic to the line segment, $\pi^{-1}(x) \cong L$. This leads to a natural way to define local coordinates on Mo, namely for every open subset U of S^1, we can define a diffeomorphism

$$\phi : U \times L \to \pi^{-1}(U) \,. \tag{2}$$

This means that to every element p of $\pi^{-1}(U) \subset Mo$, we can assign local coordinates $\phi^{-1}(p) = (x,\, t)$, where $x = \pi(p) \in U$ by definition and $t \in L$. Now, how can we quantify the non-triviality of the Möbius strip? For this, cover the circle by two open sets, U_1 and U_2, which overlap on two disjoint open intervals, A and B. We also have the diffeomorphisms

$$\begin{aligned} \phi_1 &: U_1 \times L \to \pi^{-1}(U_1) \,, \\ \phi_2 &: U_2 \times L \to \pi^{-1}(U_2) \,. \end{aligned} \tag{3}$$

It is clear that the non-triviality of Mo will reside in the way in which the different copies of L will be mapped to each other on A and B. To this end, we need an automorphism of L over the region $A \cup B = U_1 \cap U_2$. This is provided by ϕ_1 and ϕ_2 of eq. (3). For every $x \in U_1 \cap U_2$, we can define

$$\phi_1^{-1} \circ \phi_2 : (A \cup B) \times L \to (A \cup B) \times L \,. \tag{4}$$

This induces a diffeomorphism g_{12} from L to L in such a way that

$$\phi_1^{-1}\phi_2(x,\, t) = (x,\, g_{12}(t)) \,. \tag{5}$$

Since the only linear diffeomorphisms of L are the identity e and the sign-flip, $g(t) = -t$, we necessarily have that $g_{12} \in \{e,\, g\}$. We can always choose $g_{12} = e$ on A, so that for the Möbius strip g_{12} will have to equal g on B. We see that the non-triviality of the Möbius strip is encoded in the non-triviality of these 'transition functions'. We can now also understand the difference with the cylinder in another way. The same construction for the cylinder would lead to the identity on both A and B. So we see that the triviality of the cylinder is reflected in the triviality of the transition functions and that these functions encode the non-triviality of the Möbius strip. Since $g^2 = e$, in this example the transition functions form the group $\mathbb{Z}_2$. In general this group will be called the structure group of the bundle and it will turn out to be an ingredient of utmost importance in the description of bundles.

1.2 Definition of a bundle

Let us now turn to the formal definition of a fibre bundle. Most of the ingredients should now be intuitively clear from the previous example.

Definition 1. A (differentiable) **fibre bundle** $(E,\, \pi,\, M,\, F,\, G)$ consists of the following elements:

(i) A differentiable manifold E called the **total space**.

(ii) A differentiable manifold M called the **base space**.

(iii) A differentiable manifold F called the (typical) **fibre**.

(iv) A surjection $\pi : E \to M$ called the **projection**. For $x \in M$, the inverse image $\pi^{-1}(x) \equiv F_x \cong F$ is called the fibre at x.

(v) A (Lie) Group G called the **structure group**, which acts on the fibre on the left.

(vi) An open covering $\{U_i\}$ of M and a set of diffeomorphisms $\phi_i : U_i \times F \to \pi^{-1}(U_i)$ such that $\pi\phi_i(x,\, t) = x$. The map ϕ_i is called a **local trivialization**.

(vii) At each point $x \in M$, $\phi_{i,x}(t) \equiv \phi_i(x,\, t)$ is a diffeomorphism, $\phi_{i,x} : F \to F_x$. On each overlap $U_i \cap U_j \neq \emptyset$, we require $g_{ij} = \phi_{i,x}^{-1}\phi_{j,x} : F \to F$ to be an element of G, i.e. we have a smooth map $g_{ij} : U_i \cap U_j \to G$ such that

$$\phi_j(x,\, t) = \phi_i(x,\, g_{ij}(x)t).$$

In the mathematical literature this defines a coordinate bundle. Of course the properties of a bundle should not depend on the specific covering of the base manifold or choice of local trivialisations. A bundle is therefore defined as an equivalence class of coordinate bundles[4]. Since in practical applications physicists always work with an explicit choice of covering and trivialisations, we will not bother to make this distinction here.

Intuitively, one can view a fibre bundle as a manifold M with a copy of the fibre F at every point of M. The main difference with a product manifold (trivial bundle) is that the fibres can be 'twisted' so that the global structure becomes more intricate than a product. This 'twisting' is basically encoded in the transition functions which glue the fibres together in a non-trivial way. For the Möbius strip in the previous section, Mo was the total space, the base space was a circle and the fibre was the line segment L. In that example the structure group turned out to be the discrete group $\mathbb{Z}_2$. In that respect this was not a typical example, because in what follows all other examples will involve continuous structure groups. For convenience we will sometimes use $E \xrightarrow{\pi} M$ or simply E, to denote $(E,\, \pi,\, M,\, F,\, G)$.

Let us look at some of the consequences of the above definition. From (vi) it follows that $\pi^{-1}(U_i)$ is diffeomorphic to a product, the diffeomorphism given by $\phi_i^{-1} : \pi^{-1}(U_i) \to U_i \times F$. It is in this sense that E is locally a product. From their definition (vii), it is clear that on triple overlaps, the transition functions obey

$$g_{ij}g_{jk} = g_{ik}\,, \qquad \text{on} \quad U_i \cap U_j \cap U_k \neq \emptyset. \tag{6}$$

Taking $i = k$ in the above equation shows that

$$g_{ij}^{-1} = g_{ji}\,, \qquad \text{on} \quad U_i \cap U_j \neq \emptyset. \tag{7}$$

These conditions evidently have to be fulfilled to be able to glue all local pieces of the bundle together in a consistent way. A fibre bundle is trivial if and only if all transition functions can be chosen to be identity maps. Since a choice of local trivialization ϕ_i results in a choice of local coordinates, the transition functions are nothing but a transformation of 'coordinates' in going from one open subset to another. When we will discuss gauge theories they will represent gauge transformations in going from one patch to another.

Of course, one should be able to change the choice of local trivializations (coordinates) within one patch. Say that for an open covering U_i of M we have two sets

[4] For more details, see for example [8].

of trivializations $\{\phi_i\}$ and $\{\widetilde{\phi}_i\}$ of the same fibre bundle. Define a map $f_i : F \to F$ at each point $x \in U_i$

$$f_i(x) = \phi_{i,x}^{-1}\widetilde{\phi}_{i,x}. \tag{8}$$

It is easy to show that the transition functions corresponding to both trivializations are related by

$$\widetilde{g}_{ij}(x) = f_i(x)^{-1} g_{ij}(x) f_j(x). \tag{9}$$

While the g_{ij} will be gauge transformations for gluing patches together, the f_i will be gauge transformations within a patch. From eq. (9) it's clear that in general the transition functions of a trivial bundle will have the factorized form

$$g_{ij}(x) = f_i(x)^{-1} f_j(x). \tag{10}$$

1.3 More examples: vector and principal bundles

The prototype of a fibre bundle is the tangent bundle of a differentiable manifold. As described in subsection A.1, the collection of all tangent vectors to a manifold M at a point x is a vector space called the tangent space T_xM. The collection $\{T_xM|x \in M\}$ of all tangent spaces of M is called the tangent bundle TM. Its base manifold is M and fibre $\mathbb{R}^m$, where m is the dimension of M. Its structure group is a subgroup of $GL(m, \mathbb{R})$. Let us look at some examples of tangent bundles.

$T\mathbb{R}^n$. If M is $\mathbb{R}^n$, the tangent space to every point is isomorphic to M itself. Its tangent bundle $T\mathbb{R}^n$ is clearly trivial and equal to $\mathbb{R}^n \times \mathbb{R}^n \cong \mathbb{R}^{2n}$. It can be proven that every bundle over a manifold that is contractible to a point is trivial.

TS^1. The circle is not contractible, yet its tangent bundle TS^1 is trivial. The reason is that since one can globally define a (unit) vector along the circle in an unambiguous and smooth way, it is easy to find a diffeomorphism from TS^1 to $S^1 \times \mathbb{R}$.

TS^2. The tangent bundle of the 2-sphere TS^2 is our second example of a non-trivial bundle. There is no global diffeomorphism between TS^2 and $S^2 \times \mathbb{R}^2$, since to establish this one would have to be able to define two linearly independent vectors at every point of the sphere in a smooth fashion. (This is needed to be able to define coordinates on the tangent plane in a smooth way along the sphere.) In fact it's even worse for the 2-sphere, since in this case one cannot even find a single global non-vanishing vector field. The fact that this isn't possible has become known as the expression: "You cannot comb hair on a sphere."

FS^2. A set of pointwise linearly independent vectors over an open set of the base manifold of a tangent bundle is called a frame. So in the example above, the non-triviality of TS^2 was a consequence of not being able to find a frame over the entire sphere in a consistent way. At each point one can of course construct many different sets of linearly independent vectors. These are all related to each other by a transformation in the structure group, $GL(2, \mathbb{R})$ in this case. Since the action of $GL(2, \mathbb{R})$ on the set of frames is free (no fixed points for $g \neq e$) and transitive (every frame can be obtained from a fixed reference frame by a group element), the set of all possible frames over an open set U of S^2 is diffeomorphic to $U \times GL(2, \mathbb{R})$. Globally this becomes a bundle over S^2 with fibre $GL(2, \mathbb{R})$ and is called the frame bundle FS^2 of S^2. Note that for this bundle the fibre equals the structure group!

The first three examples above are examples of vector bundles, so let us define these properly.

Definition 2. A **vector bundle** $E \xrightarrow{\pi} M$ is a fibre bundle whose fibre is a vector space. If $F = \mathbb{R}^n$ it is common to call n the fibre dimension and denote it by $\dim E$ (although the total dimension of the bundle is $\dim M + n$). The transition functions belong to $GL(n, \mathbb{R})$.

Clearly a tangent bundle is always a vector bundle. Once one defines a frame $\{e_a\}$, $a \in \{1, ..., n\}$, over a patch $U \subset M$, one can expand any vector field $V : U \to \mathbb{R}^n$ over U in terms of this frame, $V = V^a e_a$. A possible local trivialization would then become

$$\phi^{-1}(p) = (\pi(p), \{V^a\})\,, \qquad p \in E \tag{11}$$

Consider two coordinate frames, associated to a set of coordinates $\{x^a\}$ on U_x and $\{y^a\}$ on U_y respectively. A vector field V on the overlap $U_x \cap U_y$ can be expanded using either frame (this notation is discussed in more detail in subsection A.1),

$$V = V^a \frac{\partial}{\partial x^a} = \widetilde{V}^a \frac{\partial}{\partial y^a} \tag{12}$$

The resulting trivializations are related as follows (in a hopefully obvious notation),

$$\phi_y^{-1} \phi_x(\pi(p),\, \{V^a\}) = (\pi(p), \{\widetilde{V}^a\}), \tag{13}$$

where

$$V^a = \frac{\partial x^a}{\partial y^b} \widetilde{V}^b. \tag{14}$$

This relation is of course well known from basic tensor calculus. In the language we developed in the previous section this would be written as

$$(g_{yx})^a{}_b = \frac{\partial x^a}{\partial y^b}. \tag{15}$$

In the frame bundle example above, the fibre was not a vector space, but a Lie group. More importantly, the fibre equalled the structure group. This is an example of a principal bundle, the most important kind of bundle for understanding the topology of gauge theories. We will need them a lot in this set of lectures, so let us look at them in a little more detail.

Definition 3. A **principal bundle** has a fibre which is identical to the structure group G. It is usually denoted by $P(M, G)$ and called a G-bundle over M.

Most of the time, G will be a Lie group. The only example we will encounter where this is not the case, is the Möbius strip. The action of the structure group on the fibre now simply becomes left multiplication within G. In addition, we can also define right multiplication (on an element p of P) as follows. Let $\phi_i : U_i \times G \to \pi^{-1}(U_i)$ be a local trivialization,

$$\phi_i^{-1}(p) = (x,\, g_i), \qquad x = \pi(p). \tag{16}$$

Right multiplication by an element a of G is defined by

$$p\, a = \phi_i(x,\, g_i a). \tag{17}$$

Since left and right multiplications commute (associativity of the group), this action is independent of the choice of local coordinates. Let $x \in U_i \cap U_j$, then

$$p\, a = \phi_j(x,\, g_j a) = \phi_j(x,\, g_{ji}(x) g_i a) = \phi_i(x,\, g_i a). \tag{18}$$

We can thus just as well write the action as $P \times G \to P : (p,\, a) \mapsto p\, a$, without reference to local choices. One can show that this action is transitive and free on $\pi^{-1}(x)$ for each $x \in M$.

1.4 Triviality of a bundle

Later on, we will discuss a number of ways in which one can quantify the non-triviality of a given bundle. Of course before we try to quantify how much it deviates from triviality, it is interesting to know wether it is non-trivial at all. We will now discuss some equivalent ways to see wether a bundle is a global product or not. Before we do that, we define one more important notion

Definition 4. A **section** s is a smooth map $s : M \to E$ such that $\pi s(x) = x$ for all $x \in M$. This is sometimes also referred to as a global section. If a section can only be defined on an open set U of M, it is called a **local section** and one only has the smooth map $s : U \to E$. The set of all sections of E is called $\Gamma(M, E)$, while the set of all local sections over U will be called $\Gamma(U, E)$.

The best known example of this is a vector field over a manifold M, which is a section of the tangent bundle TM. Clearly, it is not a great challenge to construct a local section over some open subset $U \subset M$. Being able to construct a (global) section over M will place much stronger requirements on the topology of the bundle and it will have a lot to say about the non-triviality of a bundle. This is reflected in the following theorem:

Theorem 1. A vector bundle of rank n is trivial if and only if it admits n point-wise linearly independent sections, i.e. a global frame.

This is precisely why the tangent bundle over the 2-sphere is non-trivial. On the other hand, for a vector bundle there always exists at least one global section, namely the so called zero section, i.e. the section which maps every point $x \in M$ to (x, O), where O is the origin of the vector space. This is always possible irrespective of the non-triviality of the bundle, since we do not need a frame to characterize the origin uniquely. This is not so for a principal bundle and the group structure of the fibre allows for the following very powerful theorem:

Theorem 2. A principal bundle is trivial if and only if it allows a global section.

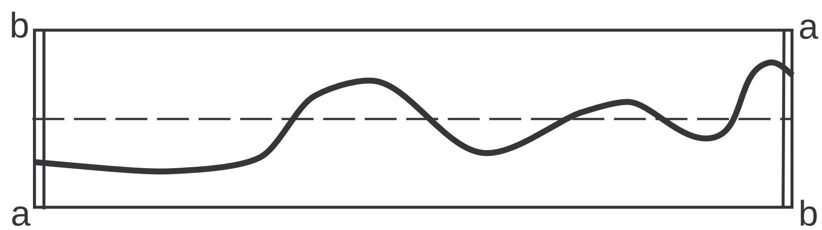

Figure 2: A section of the bundle corresponding to the Möbius strip. The points marked with the same letter (a or b) should be identified.

To illustrate the above theorems, we return to the discussion of the Möbius strip Mo. In the first section we saw that the non-triviality of the Möbius strip had to do with the fact that one could not find a global trivialization. We now understand that this is the case because one cannot define a linearly independent (which in one dimension means everywhere nonzero) section on Mo[5]. It is not very difficult to see

[5] The reader might be bothered by the fact that the Möbius strip is not a vector bundle. One can however replace the line segment L by the real line $\mathbb{R}$ and all the arguments used in the text will still apply. In this case one would speak of a line bundle over S^1

(figure 2) that every smooth section would have to take the value zero over at least one point of the circle. According to Theorem 1 this is equivalent to the bundle being non-trivial.

To illustrate the second theorem, we would like to associate a principal bundle $P(S^1, \mathbb{Z}_2)$ to Mo. To accomplish this, replace the fibre L by the structure group $\mathbb{Z}_2$. Take the same open covering of the circle as in subsection 1.1 and use the same transition functions on the overlap $A \cup B$, where now instead of acting on the line segment L, they act by left multiplication within $\mathbb{Z}_2$. What one gets is a double cover of the circle as depicted in figure 3. To get a global section one has to go around the circle once and it is clear that this will always have a discontinuity somewhere. Thus we cannot find a section of this principal bundle, so that according to Theorem 2 it has to be non-trivial.

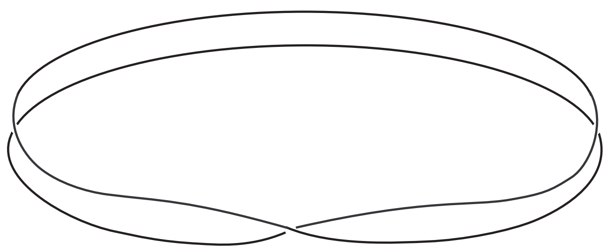

Figure 3: The principal bundle $P(S^1, \mathbb{Z}_2)$ associated to the Möbius strip is a double cover of the circle.

The above construction is more general: starting from a fibre bundle (not necessarily a vector bundle), one can always construct the associated principal bundle by replacing the fibre with the structure group and keeping the transition functions. Note that the frame bundle FS^2 over S^2 from subsection 1.3 was nothing but the principal bundle associated to the tangent bundle TS^2. More generally two bundles with same base space and structure group are called associated if their respective associated principal bundles are equivalent. In gauge theories it is important to be able to associate a principal bundle to a vector bundle and vice versa.

Definition 5. Start from a principal bundle $P(M,\, G)$ and an n-dimensional faithful representation $\rho : G \to GL(n,\, \mathbb{R})$ which acts on $V = \mathbb{R}^n$ from the left. Consider the product $P \times V$. Define the equivalence relation $(p,\, v) \sim (p\, g^{-1},\, \rho(g)v)$, where $p \in P$, $v \in V$ and $g \in G$. The vector bundle E_ρ associated to P via the representation ρ is defined as

$$E_\rho = P \times_\rho V \equiv P \times V / \sim \tag{19}$$

This is basically a complicated way of saying that one changes the fibre from G to V and use as transition function $\rho(g_{ij})$ instead of g_{ij}. Since every element of P over a certain point $x \in M$ can be obtained from $(x,\, e)$ (where e is the identity of G) by an element of G, the equivalence relation $(p\, g,\, v) \sim (p,\, \rho(g)v)$ effectively replaces the fibre over x with V, thus replacing a principal bundle by a vector bundle. If we define the new projection by

$$\pi_E[(p,\, v)] = \pi(p), \tag{20}$$

this is well defined under the equivalence relation since $\pi(p\, g) = \pi(p)$. If $\phi_P(\pi(p),\, g) = p$, $p \in P$ is a local trivialization on $U \subset M$, we define for E_ρ

$$\phi_E^{-1} : E_\rho \to U \times V : [(p,\, v)] \mapsto (\pi(p),\, \rho(g)v). \tag{21}$$

This definition is independent of the representative of the equivalence class. To see this, take two different representatives:

$$[(p,\, v)] = [(p\, h^{-1},\, \rho(h)v)]. \tag{22}$$

To find their trivialization corresponding to a trivialization over U in the associated principal bundle P, we note that

$$p = \phi_P(x, g), \qquad x = \pi(p); \tag{23}$$

$$p\, h^{-1} = \phi_P(x, g\, h^{-1}), \qquad x = \pi(p\, h^{-1}) = \pi(p). \tag{24}$$

From this, we find

$$[(p\, h^{-1},\, \rho(h)v)] = \phi_E(x,\, \rho(g\, h^{-1})\rho(h)v) = \phi_E(x,\, \rho(g)v). \tag{25}$$

To see what the transition functions are, consider a point $[(p,\, v)] \in E_\rho$. If we choose a trivialization of U_i such that $p = \phi^i_P(x,\, e)$, then on U_j there is a trivialization such that on $U_i \cap U_j \neq \emptyset$, we have $p = \phi^i_P(x,\, g_{ji})$. For the corresponding trivializations of E_ρ, this means

$$[(p,\, v)] = \phi^i_E(x, v) = \phi^j_E(x, \rho(g_{ji})v) = \phi^j_E(x, \rho_{ji}v). \tag{26}$$

This shows that the new transition functions ρ_{ij} are just $\rho(g_{ij})$.

Now that we have defined all necessary ingredients, we can formulate the main theorem of this subsection[6]

Theorem 3. A vector bundle is trivial if and only if its associated principal bundle is trivial.

This means that for establishing the (non-)triviality of a vector bundle, we only need to study its associated principal bundle. More concretely:

Corollary 1. A vector bundle is trivial if an only if its associated principal bundle admits a section.

Sometimes the converse is more useful:

Corollary 2. A principal bundle is trivial if an only if its associated vector bundle of rank n admits n point-wise linearly independent sections.

This is exactly why both the bundle Mo and its associated principal bundle were non-trivial. We were basically looking at the same issue from two different points of view.

2 Connections on fibre bundles

Now that we have gained some feeling for the concept of a bundle, we want to define some extra structure on it. We are all familiar with the notion of parallel transport of a tangent vector on a manifold in General Relativity. Given a curve in space-time, there are many possible choices to transport a given vector along this curve, which are all equally valid as a choice for what 'parallel' might mean[7]. Translated into the language of fibre bundles, we want to, given a curve γ in the base (space-time)

[6]There is actually a more general theorem which states that a bundle is trivial iff all its associated bundles are trivial.

[7]The Levi-Civita connection corresponds to a choice of parallel transport such that if a curve is the shortest path between two points, a tangent vector to this curve stays tangent to the curve under parallel transport.

M, define a corresponding section of the tangent bundle $s^\gamma \in TM$, in such a way that $\pi(s^\gamma) = \gamma$ (otherwise it would not be a section).

The question we want to ask ourselves now is basically a generalization of this. Given a certain motion in the base manifold, how can we define a corresponding motion in the fibre? Since, given a bundle, there is no a priori notion of what 'parallel' should mean, we need some additional structure to give meaning to the notion of 'parallel motion'. This will be provided by the choice of a connection on the bundle. This section follows parts of [1] and [4] closely, although some specific points are more indebted to [3]. We will start by defining parallel transport on a principal bundle and later sketch how this connection can be used to provide a connection on an associated vector bundle. But first of all, let us pause and recall some facts about Lie groups.

2.1 Lie groups and algebras

From now on the structure group G will be a Lie group, i.e. a differential manifold with a group structure, where the group operations (multiplication, inverse) are differentiable. Given an element $g \in G$, we can define the left- and right-translation of an element $h \in G$ by g,

$$L_g(h) = gh; \tag{27}$$
$$R_g(h) = hg. \tag{28}$$

These induce differential maps in the tangent space (see subsection A.3 for a review of differential maps)

$$L_{g*}: \qquad T_hG \to T_{gh}G; \tag{29}$$
$$R_{g*}: \qquad T_hG \to T_{hg}G. \tag{30}$$

We say that a vector field X is left-invariant if it satisfies

$$L_{g*}\left(X|_h\right) = X|_{gh}. \tag{31}$$

One can show that if X and Y are left-invariant vector fields, their Lie bracket $[X, Y]$ is also left-invariant. The algebra formed in this way by the left-invariant vector fields of G, is called the Lie algebra $\mathfrak{g}$. Because of (31), a vector A in T_eG (e is the unit element of G) uniquely defines a left-invariant vector field X_A over G (that is, a section of TG). This establishes an isomorphism between T_eG and $\mathfrak{g}$. From now on we will not make the distinction between the two anymore and say that the Lie algebra $\mathfrak{g}$ is the tangent space to the identity in G. The generators T_a, $a \in \{1, ..., r\}$ ($r = \dim\mathfrak{g} = \dim G$) of $\mathfrak{g}$ satisfy the well known relations

$$[T_a, T_b] = f_{ab}{}^c\, T_c\,, \tag{32}$$

where $f_{ab}{}^c$ are the structure constants of $\mathfrak{g}$ (and can be shown to really be constant).

By Lie group we will always mean a matrix group (subgroup of $GL(n)$ with matrix multiplication as group operation), although most of what we will discuss can be proven in a more general context. From our experience with matrix groups we know that exponentiation maps elements of the Lie algebra to elements of the Lie group. Consequently, $A \in \mathfrak{g}$ generates a curve (one-parameter subgroup) through g in G by

$$\sigma_t(g) = g\exp(tA) = R_{\exp(tA)}(g). \tag{33}$$

The corresponding flow equation is (in matrix notation)

$$\left.\frac{d\sigma_t(g)}{dt}\right|_{t=0} = gA = L_{g*}A = X_A|_g\,, \tag{34}$$

where we used that in matrix notation $L_{g*}A = gA$ (exercise) and that A generates a left-invariant vector field X_A. This shows that the tangent vector to the curve through g is nothing but the left-translation of A by L_{g*} (or the left-invariant vector field generated by A evaluated at g, $X_A|_g$). More formally one would define the tangent vector to the one-parameter flow by making use of a function $f : G \to \mathbb{R}$,

$$X_A(f(g)) = \frac{d}{dt} f(\sigma_t(g))\bigg|_{t=0}. \tag{35}$$

Given a basis $\{T_a\}$ of $\mathfrak{g}$, one defines a corresponding basis $\{X_a\}$ of T_gG by left translation to $g \in G$, $X_a = L_{g*}T_a$. This means that for an element $A \in \mathfrak{g}$, if $A = A^a T_a$, the left-invariant vector field corresponding to A can be expanded as $X_A = A^a X_a$. One can also define a basis for left-invariant 1-forms $\{\eta^a\}$ dual to to $\{X_a\}$ by, $\eta^a(X_b) = \delta^a_b$. The Maurer-Cartan form Θ is then defined by

$$\Theta = T_a \otimes \eta^a. \tag{36}$$

This is a $\mathfrak{g}$-valued 1-form, which takes a left-invariant vector field $X|_g$ at g and pulls it back to the identity, giving back the Lie algebra element $X|_e$. To see this, evaluate the Maurer-Cartan form on X_A defined above,

$$\Theta(X_A)|_g = T_a \otimes \eta^a(A^b X_b) = A^a T_a = A. \tag{37}$$

In this way, the Maurer-Cartan form establishes an explicit isomorphism between $\mathfrak{g}$ and T_eG. More generally, it takes any vector at g and returns a Lie algebra element, thus establishing the decomposition of the vector in terms of a basis of left-invariant vector fields at g.

To make contact with widely used notation in physics, we will now be a bit less formal. Choosing coordinates $\{g^i\}$ on a patch of G, a coordinate basis at g can be written as $\{\partial/\partial g^i\}$. A coordinate basis at the identity e would in this notation be written as

$$\frac{\partial}{\partial g^i}\bigg|_e = L_{g^{-1}*}\frac{\partial}{\partial g^i}. \tag{38}$$

This would mean that (36) can be rewritten as

$$\Theta = L_{g^{-1}*}\frac{\partial}{\partial g^i} \otimes dg^i = g^{-1}\frac{\partial}{\partial g^i} \otimes dg^i, \tag{39}$$

where the last equality is for a matrix group. Physicists write this as

$$\Theta = g^{-1}dg, \tag{40}$$

where dg should be interpreted as the identity operator at g,

$$dg = \frac{\partial}{\partial g^i} \otimes dg^i. \tag{41}$$

Since for a matrix group $X_A|_g = gA$, we get indeed that

$$\Theta(X_A) = g^{-1}dg(X_A) = g^{-1}gA = A. \tag{42}$$

The reason why this notation makes sense is because if A is tangent to a flow $\sigma_t(e)$, $X_A|_g = gA$ is tangent to the flow $\sigma_t(g)$. Concretely, we have

$$A = \frac{d\sigma_t(e)}{dt}\bigg|_{t=0}, \tag{43}$$

so that,

$$X_A|_g = gA = \left.\frac{d\sigma_t(g)}{dt}\right|_{t=0} \equiv X_A(g) = dg(X_A). \tag{44}$$

We see that if we interpret $X_A(g)$ as $X_A|_g$ (for a matrix group in matrix notation), we can almost act as if $dg(X_A)$ has the usual meaning. In a lot of practical calculations (40) is used in an even more direct sense: if $g = \exp tA$ for some $A \in G$ and some t,

$$\Theta|_g = h^{-1}dh|_g \equiv e^{-tA}\left.\frac{d}{dt'}e^{t'A}\right|_{t'=t} = A, \tag{45}$$

where $\Theta|_g$ should now be thought of as simply the Lie algebra element associated to g. What we are doing here is exactly the same as above, because

$$\left.\frac{d}{dt'}e^{t'A}\right|_{t'=t} = e^{tA}A = gA = g\left.\frac{d}{dt'}e^{t'A}\right|_{t'=0}, \tag{46}$$

so that

$$\Theta|_g = e^{-tA}\left.\frac{d}{dt'}e^{t'A}\right|_{t'=t} = g^{-1}\left.\frac{d}{dt'}ge^{t'A}\right|_{t'=0} = \Theta(X_A), \tag{47}$$

where X_A is the left invariant vector field associated to A.

Finally the adjoint map

$$ad_g : G \to G : h \mapsto ghg^{-1}, \tag{48}$$

induces a map in the tangent space

$$ad_{g*} : T_hG \to T_{ghg^{-1}}G. \tag{49}$$

When we restrict this map to $h = e$, we get the adjoint representation of $T_eG \cong \mathfrak{g}$,

$$Ad_g : \mathfrak{g} \to \mathfrak{g} : V \mapsto gVg^{-1}. \tag{50}$$

2.2 Parallel transport in a principal bundle

Consider a principal bundle $P(M, G)$. Given a curve γ on the base manifold M, to define parallel transport, we want to define a corresponding choice of curve γ_P in the total space P. There are of course many possible choices. To characterize our choice, we will look at vectors tangent to this curve. At every point along γ, we can define a lift of the tangent vector to γ to an element in TP, the tangent vectors to P. This will define an integral curve γ_P in P. See figure 4 for an illustration

The question is then how to lift a vector in $T_\gamma M$ to TP. At every point $p \in P$, we can decompose T_pP into a subspace of vectors tangent to the fibre G, called the vertical subspace V_pP and a complement H_pP, called the horizontal subspace, such that $T_pP = V_pP \oplus H_pP$. Since V_pP corresponds to motion along the fibres and is essentially fixed, a choice of H_pP is the crucial ingredient in the definition of parallel transport. We will require the vectors tangent to γ_P to lie in H_pP.

A choice of connection now essentially boils down to a choice of horizontal subspace. Let us be a bit more precise.

Definition 6. A **connection** on P is a smooth and unique separation of the tangent space T_pP at each p into a vertical subspace V_pP and a horizontal subspace H_pP such that

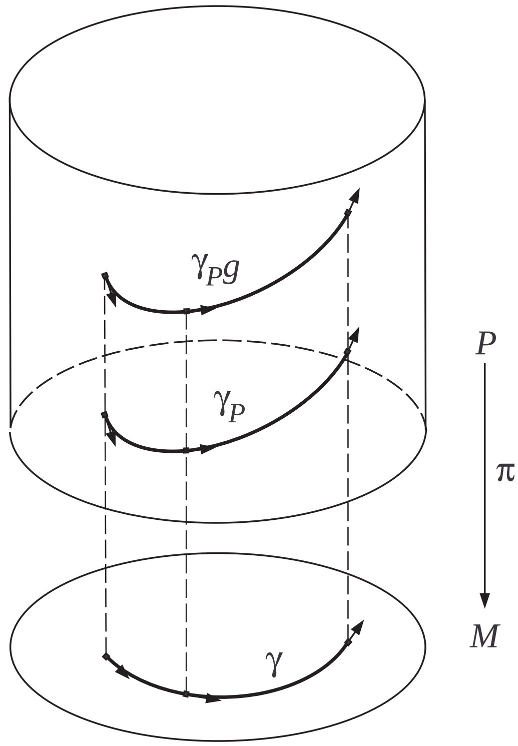

Figure 4: Illustration of horizontal lift.

(i) $T_p P = V_p P \oplus H_p P$;

(ii) $H_{pg} P = R_{g*} H_p P$ for every $g \in G$.

Condition (i) just means that every $X \in T_p P$ can be written in a unique way as a sum $X = X^V + X^H$, where $X^V \in V_p P$ and $X^H \in H_p P$. The equivariance condition (ii) means that the choice of horizontal subspace at p determines all the horizontal subspaces at points pg. This roughly means that all points above the same point $x = \pi(p)$ in the base space will be parallel transported in the same way (recall that $\pi(p) = \pi(pg)$).

Parallel transport can now immediately be defined by what is called a horizontal lift.

Definition 7. Let $\gamma : [0, 1] \to M$ be a curve in the base manifold (a base curve). A curve $\gamma_P : [0, 1] \to P$ is called the **horizontal lift** of γ if

(i) $\pi(\gamma_P) = \gamma$;

(ii) All tangent vectors X_P to γ_P are horizontal: $X_P \in H_{\gamma_P} P$.

Theorem 4. Let $\gamma : [0, 1] \to M$ be a base curve and let $p \in \pi^{-1}(\gamma(0))$. Given a connection, there exists a unique horizontal lift γ_P such that $\gamma_P(0) = p$.

This means that we can (given a connection) uniquely define the parallel transport of a point p in P along a curve γ in M by moving it along the unique horizontal lift of γ through p.

A loop in M is defined as a curve γ with $\gamma(0) = \gamma(1)$. It is interesting to see what happens to a horizontal lift of this loop. In other words, what happens if we parallel transport an element of P along a closed loop? Starting from a point p and moving it along a horizontal lift of a loop, there is no guarantee that we will end up at the same point. In general we will obtain a different point p_γ, which depends on the loop γ. Since

$$\pi(\gamma_P(0)) = \gamma(0) = \gamma(1) = \pi(\gamma_P(1)), \tag{51}$$

we know that both points will belong to the same fibre, $\pi(p) = \pi(p_\gamma)$. This means that $p_\gamma = pg$ for some $g \in G$. If we vary the loop γ, but keep the base point p fixed, we generate a group called the holonomy group $Hol_p(P)$ of P at p, which by definition is a subgroup of G. This group of course also depends on the connection, so $Hol_p(P)$ is a characteristic not only of P, but also of the connection. If M is connected the holonomy group at all points of M are isomorphic and we can speak of the holonomy group of P, $Hol(P)$.

Given a notion of parallel transport in a principal bundle P, one can easily define parallel transport in an associated bundle E_ρ by

Definition 8. If $\gamma_P(t)$ is a horizontal lift of $\gamma(t) \in M$ in P, then $\gamma_E(t)$ is defined to be the horizontal lift of $\gamma(t)$ in E_ρ if

$$\gamma_E(t) = [(\gamma_P(t), v)], \tag{52}$$

where v is a constant element of V.

This is independent of the lift chosen in P, since (because of equivariance) another lift would be related to γ_P by $\gamma'_P(t) = \gamma_P(t)a$, with a constant element $a \in G$, so that

$$\gamma'_E(t) = [(\gamma'_P(t), v)] = [(\gamma_P(t), \rho(a)^{-1}v)] \tag{53}$$

where $\rho(a)^{-1}v$ is still a constant element. So this would still be a horizontal lift, albeit through a different element of V. Choosing a trivialization for γ_P, $\gamma_P(t) = \phi_P(\gamma(t), g(t))$, leads to the corresponding trivialization for γ_E

$$\gamma_E(t) = \phi_E(\gamma(t), \rho(g(t))v). \tag{54}$$

We see that if parallel transport in P is described by $g(t)$, then parallel transport in E_ρ is defined by $\rho(g(t))$.

2.3 Connection one-form on a principal bundle

Up to this point, the reader might be confused as to what this all has to do with gauge theories and the usual definition of a connection in physics. To establish the link with physics, we now introduce the connection one-form and clarify its relation to the gauge potential in Yang-Mills theories.

To define the connection one-form properly, we need a more specific construction of the vertical subspace V_pP. Let $A \in \mathfrak{g} = T_eG$ be an element of the Lie algebra of G. We saw in section 2.1 that A generates a one-parameter flow $\sigma_t(g)$ through g in G. A slight modification of this construction shows that A will generate a flow in P along the fibre at each point of M by the right action of G on P:

$$\sigma_t(p) = R_{\exp(tA)}p = p\,\exp(tA). \tag{55}$$

Note that $\pi(p) = \pi(\sigma_t(p))$, so that indeed vectors tangent to the curves are elements of V_pP. We now define a map $\mathfrak{g} \to V_pP$ which maps A to the vector tangent to

$\sigma_t(p)$ for $t = 0$, which we will call (with slight abuse of notation with respect to equation (34)) $X_A \in V_pP$. The equivalent of the flow equation (35) now becomes

$$X_A(f(p)) = \left.\frac{d}{dt} f(\sigma_t(p))\right|_{t=0} . \tag{56}$$

X_A is called the fundamental vector field associated with A. The fundamental vector fields associated to a basis of the Lie algebra form a basis of the vertical subspace. A connection one-form is now defined as follows.

Definition 9. A **connection one-form** $\omega \in \Lambda P \otimes \mathfrak{g}$ (where $\Lambda P \equiv \Gamma(P, T^*P)$) is a Lie algebra valued one-form defined by a projection of the tangent space T_pP onto the vertical subspace V_pP satisfying

(i) $\omega(X_A) = A$ for every $A \in \mathfrak{g}$;

(ii) $R_g^*\omega = Ad_{g^{-1}}\omega$ for every $g \in G$.

More concretely, (i) means that ω acts as a Maurer-Cartan form on the vertical subspace and (ii) means that for $X \in T_pP$,

$$R_g{}^*\omega|_p(X) = \omega|_{pg}(R_{g*}X) = g^{-1}\omega(X)g. \tag{57}$$

The horizontal subspace is then defined as the set

$$H_pP = \{X \in T_pP |\, \omega(X) = 0\}. \tag{58}$$

When defined in this way, H_pP still satisfies the equivariance condition. To see this, take $X \in H_pP$ and construct $R_{g*}X \in T_{pg}P$. This is an element of H_{pg} because

$$\omega(R_{g*}X) = R_g{}^*\omega(X) = g^{-1}\omega(X)g = 0. \tag{59}$$

So both definitions of a connection are equivalent. This connection is defined over all of P. To connect to physics, we have to relate this to a one-form in the base M. It turns out that this can only be done locally (when P is non-trivial).

Definition 10. Let $\{U_i\}$ be an open covering of M. Choose a local section s_i on U_i

$$s_i : U_i \to \pi^{-1}(U_i). \tag{60}$$

The local connection one-form or **gauge potential** is now defined as

$$A_i \equiv s_i{}^*\omega \in \Gamma(U_i, T^*M,) \otimes \mathfrak{g} = \Lambda U_i \otimes \mathfrak{g} \tag{61}$$

Also the converse is true.

Theorem 5. Given a local connection one-form A_i and a section s_i on an open subset $U_i \subset M$, there is a unique connection one-form $\omega \in \pi^{-1}(U_i)$ such that $A_i = s_i{}^*\omega$.

We will prove this theorem rather explicitly, since this will give better insight into the emergence of a gauge potential on M from the connection one-form on P. First of all, we introduce the notion of a canonical local trivialization with respect to a section. Given a section s_i on U_i and a $p \in \pi^{-1}(U_i)$, there always exists a $g_i \in G$ such that $p = s_i(x)g_i$, where $x = \pi(p)$. This means that we can define a local trivialization by

$$\phi^{-1} : \pi^{-1}(U_i) \to U_i \times G : p \mapsto (x,\, g_i). \tag{62}$$

This means that the section itself is represented as $s_i(x) = (x, e)$. On an overlap $U_i \cap U_j \neq \emptyset$ two sections are related by

$$\begin{aligned} s_i(x) &= \phi_i(x, e) = \phi_j(x, g_{ji}(x)e) = \phi_j(x, g_{ji}(x)) \\ &= \phi_j(x, e)g_{ji}(x) = s_j(x)g_{ji}(x). \end{aligned} \tag{63}$$

First we proof that an ω exists, then we will sketch a proof of its uniqueness.

Proof (existence):
Given a section s_i and a gauge potential A_i on U_i, we propose the following form of the connection one-form:

$$\omega|_{U_i} = g_i^{-1}\pi^* A_i g_i + g_i^{-1} d_P g_i \,, \tag{64}$$

where d_P is the exterior derivative on P and g_i is the group element which appears in the definition of the canonical local trivialization with respect to s_i.

(i) First of all, we have to show that pulling back (64) with the section results in A_i. Note that since $\pi \circ s_i = \mathrm{Id}_{U_i}$, we have that $\pi_* s_{i*} = \mathrm{Id}_{TU_i}$ and that $g_i = e$ on s_i. For a $X \in T_xM$ we have

$$\begin{aligned} {s_i}^*\omega|_{U_i}(X) &= \omega(s_{i*}X) = \pi^* A_i(s_{i*}X) + d_P e(s_{i*}X) \\ &= A_i(\pi_* s_{i*}X) = A_i(X). \end{aligned} \tag{65}$$

(ii) Now we need to establish that (64) satisfies the conditions from definition 9. A fundamental vector field V_A satisfies $\pi_* X_A = 0$ so that only the second term from (64) contributes. We need to evaluate this in the sense of a Maurer-Cartan form as discussed in subsection (2.1),

$$\begin{aligned} w|_{U_i}(X_A) &= g_i^{-1} d_P g_i(X_A) = g_i^{-1} X_A(g_i) = g_i^{-1} X_A|_{g_i} \\ &= g_i^{-1}(p)\, \frac{d}{dt} g_i(\sigma_t(p))\bigg|_{t=0} = g_i^{-1}(p)\, \frac{d}{dt} g_i(p\,\exp(tA))\bigg|_{t=0} \\ &= g_i^{-1}(p) g_i(p)\, \frac{d}{dt}\exp(tA)\bigg|_{t=0} = \frac{d\sigma_t(e)}{dt}\bigg|_{t=0} = A, \end{aligned} \tag{66}$$

where we used both the definition of the fundamental vector field (56) and the flow equation (34). This proves the first condition for being a connection. To prove the second condition, take an $X \in T_pP$. Note next that $g_i(p\,h) = g_i(p)h$ and that since $\pi \circ R_h = \pi$, we have that $\pi_* R_{h*} = \pi_*$. We find

$$\begin{aligned} {R_h}^*\omega(X) &= \omega(R_{h*}X) = h^{-1} g_i^{-1} A_i(\pi_* X) g_i h + h^{-1} g_i^{-1} d_P g_i(X) h \\ &= h^{-1}\omega(X) h = Ad_{h^{-1}}\omega(X). \end{aligned} \tag{67}$$

□

We will now sketch the proof of the uniqueness of the connection one-form. For this we need to see what happens on overlaps $U_i \cap U_j \neq \emptyset$.

Proof (uniqueness):
The two definitions of the connection on each patch have to agree on the intersection, $\omega|_{U_i} = \omega|_{U_j}$ on $U_i \cap U_j \neq \emptyset$. Writing this out, we find the following condition:

$$g_i^{-1}\pi^* A_i g_i + g_i^{-1} d_P g_i = g_j^{-1}\pi^* A_j g_j + g_j^{-1} d_P g_j \tag{68}$$

Noting that on the intersection we have $g_j = g_{ji}g_i$ ($s_j = s_i g_{ij}$), a small calculation shows that

$$\pi^* A_j = g_{ij}^{-1}\pi^* A_i g_{ij} + g_{ij}^{-1} d_P g_{ij}. \tag{69}$$

We can use either one of the sections $s_{i,j}$ on $U_i \cap U_j$ to pull this back to a local statement (note that $s_i^*\pi^* = \mathrm{Id}_M$ and that the pull-back and the exterior derivative commute),

$$A_j = g_{ij}^{-1} A_i g_{ij} + g_{ij}^{-1} dg_{ij}. \tag{70}$$

So we see that both definitions of ω agree on $U_i \cap U_j \neq \emptyset$ if the local gauge potentials are related in the above way.

□

This is the important point we wanted to reach. Note that the connection one-form ω on P is defined globally; it contains global information on the non-triviality of P. The gauge potentials $\{A_i\}$ are defined locally and we have just seen that if the fibres over two intersecting open sets on the base space have to be identified in a non-trivial way, the two gauge potentials defined on the overlap are necessarily different. This means that on a non-trivial bundle one local gauge potential has no global information, only the collection of all locally defined gauge potentials knows about the global topology. This means that gauge freedom is sometimes not just a matter of choice, but more of necessity!

Of course also in this language gauge freedom is a reflection of choice. Say that on an open set U, two sections are related by $s'(x) = s(x)g(x)$. We can choose either section to define a local gauge potential and almost the same reasoning as above shows that both are related as follows

$$A'(x) = g(x)^{-1}A(x)g(x) + g(x)^{-1}dg(x). \tag{71}$$

We see that in the bundle language, gauge freedom is equivalent to the freedom to choose local coordinates on a principal bundle!

2.4 Curvature of a connection

A very important notion is of course the curvature of a connection. To introduce this, we first define the concept of covariant derivative on a principal bundle.

Definition 11. Consider a Lie algebra valued p-form, $\alpha \in \Lambda^n P \otimes \mathfrak{g}$. This can be decomposed as $\alpha = \alpha^a \otimes T_a$, where α^a is an ordinary p-form and T_a is a basis of $\mathfrak{g}$. Let $X_1, ..., X_{p+1} \in T_pP$ be $p+1$ tangent vectors on P. The **exterior covariant derivative** of α is defined by

$$D\alpha(X_1, ..., X_{p+1}) = d_P\alpha(X_1^H, ..., X_{p+1}^H), \tag{72}$$

where $X_i^H \in H_pP$ is the horizontal component of $X_i \in T_pP$ and $d_P\alpha = (d_P\alpha^a)\otimes T_a$.

The curvature is now readily defined.

Definition 12. The **curvature 2-form** Ω of a connection ω on P is defined as

$$\Omega = D\omega \in \Lambda^2 P \otimes \mathfrak{g} \tag{73}$$

At every point $p \in P$, the horizontal vectors define a subspace of T_pP. The assignment of such a subspace at every point of P is called a distribution. Since in this case the distribution is defined by the equation $\omega = 0$, the Frobenius condition for integrability of the submanifold of P tangent to this distribution is exactly the vanishing of $d\omega$ along the distribution, that is, the vanishing of the curvature. As such, the curvature is an obstruction to finding a submanifold of P that is 'completely horizontal'.

Theorem 6. The curvature 2-form has the property

$$R_g{}^*\Omega = Ad_{g^{-1}}\Omega = g^{-1}\Omega g, \tag{74}$$

where g is a constant element of G.

Proof:
Recall that, because of the equivariance property of horizontal subspaces

$$\begin{aligned}
&(R_{g*}X)^H = R_{g*}X^H; \\
&R_g{}^*\omega = Ad_{g^{-1}}\omega = g^{-1}\omega g.
\end{aligned}$$

Also recall that pull-backs and exterior derivatives commute, $d_P R_g{}^* = R_g{}^* d_P$ and because g is constant, $d_P g = 0$. For X, $Y \in T_pP$ we then have

$$\begin{aligned}
R_g{}^*\Omega(X,Y) &= \Omega(R_{g*}X, R_{g*}Y) = d_P\omega(R_{g*}X^H, R_{g*}Y^H) \\
&= R_g{}^* d_P\omega(X^H, Y^H) = d_P R_g{}^*\omega(X^H, Y^H) \\
&= d_P(g^{-1}\omega g)(X^H, Y^H) = g^{-1} d_P\omega(X^H, Y^H) g \\
&= g^{-1}\Omega(X,Y)g
\end{aligned} \tag{75}$$

□

Let $\alpha \in \Lambda^p \otimes \mathfrak{g}$ and $\beta \in \Lambda^q \otimes \mathfrak{g}$ be two Lie algebra valued forms. The Lie bracket (commutator) between the two is defined as

$$\begin{aligned}
[\alpha,\, \beta] &\equiv \alpha \wedge \beta - (-)^{pq}\beta \wedge \alpha \\
&= T_a T_b \alpha^a \wedge \beta^b - (-)^{pq} T_b T_a \beta^b \wedge \alpha^a \\
&= [T_a, T_b]\alpha^a \wedge \beta^b = f_{ab}{}^c T_c\, \alpha^a \wedge \beta^b.
\end{aligned} \tag{76}$$

Note that for odd p this means that

$$[\alpha, \alpha] = 2\alpha \wedge \alpha \neq 0. \tag{77}$$

For even p, $[\alpha, \alpha] = 0$.

Let us now proof the following important theorem:

Theorem 7. Ω and ω satisfy the Cartan structure equations (X, $Y \in T_pP$)

$$\Omega(X,Y) = d_P\omega(X,Y) + [\omega(X), \omega(Y)], \tag{78}$$

or

$$\Omega = d_P\omega + \omega \wedge \omega = d_P\omega + \frac{1}{2}[\omega, \omega]. \tag{79}$$

To see the relation between the two forms of the theorem note that

$$\begin{aligned}
[\omega, \omega](X,Y) &= [T_a, T_b]\omega^a \wedge \omega^b(X,Y) \\
&= [T_a, T_b](\omega^a(X)\omega^b(Y) - \omega^a(Y)\omega^b(X)) \\
&= [\omega(X), \omega(Y)] - [\omega(Y), \omega(X)] = 2[\omega(X), \omega(Y)].
\end{aligned}$$

Proof:
We consider three cases separately

(i) Let X, $Y \in H_pP$. Then by definition $\omega(X) = \omega(Y) = 0$. Then (78) follows trivially since by definition

$$\Omega(X,Y) = d_P\omega(X^H, Y^H) = d_P\omega(X,Y).$$

(ii) Let $X \in H_pP$ and $Y \in V_pP$. Since $Y^H = 0$, by definition $\Omega(X,Y) = 0$. We still have that $\omega(X) = 0$, so we still have to prove that $d_pP\omega(X,Y) = 0$. To do this, we use the following identity:

$$\begin{aligned} d_P\omega(X,Y) &= X\omega(Y) - Y\omega(X) - \omega([X,Y]) \\ &= X\omega(Y) - \omega([X,Y]). \end{aligned}$$

Since $Y \in V_pP$, it is a fundamental vector field[8] for some $V \in \mathfrak{g}$. This means that $\omega(Y) = V$ is a constant, so $X\omega(Y) = XV = 0$. One can show that $[X,Y] \in H_pP$, so that also $\omega([X,Y]) = 0$.

(iii) Let X, $Y \in V_pP$. Then again $\Omega(X,Y) = 0$ and this time we have

$$d_P\omega(X,Y) = -\omega([X,Y]).$$

So we still have to prove $\omega([X,Y]) = [\omega(X), \omega(Y)]$. Since also $[X,Y] \in V_pP$, every vector $X = X_B$, $Y = X_C$ and $[X,Y] = X_A$ are fundamental vector fields associated to Lie algebra elements B, C and A respectively. One can prove that necessarily $A = [B,C]$, which completes the proof.

□

We will now again use a section to pull back this globally defined object on P to a local object defined on a patch on M.

Definition 13. Given a section s_i on U_i, the local (Yang-Mills) **field strength** is defined by

$$F_i = {s_i}^*\Omega \in \Lambda^2 U_i \otimes \mathfrak{g}. \tag{80}$$

The relation with the gauge potential is now easily obtained

$$\begin{aligned} F_i &= {s_i}^* d_P\omega + {s_i}^*(\omega \wedge \omega) = d({s_i}^*\omega) + {s_i}^*\omega \wedge {s_i}^*\omega \\ &= dA_i + A_i \wedge A_i. \end{aligned} \tag{81}$$

Writing $A = A_a dx^a$ and $F = \frac{1}{2}F_{ab}dx^a \wedge dx^b$ (we dropped the subscript i for convenience), we find the usual expression,

$$F_{ab} = \partial_a A_b - \partial_b A_a + [A_a, A_b]. \tag{82}$$

The effect of a coordinate change on the field strength 2-form can be deduced (as usual) from the transformation properties of the gauge potential 1-form (71). More specifically, if two sections are related by $s'(x) = s(x)g(x)$, the corresponding field strengths are related by

$$F'(x) = g(x)^{-1}F(x)g(x). \tag{83}$$

We will need an important identity involving the curvature. Since, $\omega(X) = 0$ for $X \in H_pP$, we find that for X, Y, $Z \in T_pP$

$$D\Omega(X,Y,Z) = d_P\Omega(X^H, Y^H, Z^H) = (d_P\omega \wedge \omega - \omega \wedge d_P\omega)(X^H, Y^H, Z^H) = 0.$$

[8] Or a linear combination of fundamental vector fields, in which case the result follows by linearity

This proves the **Bianchi identity**

$$D\Omega = 0. \tag{84}$$

To find the local form of this identity we use a section s_i to pull back the relation

$$d_P\Omega = d_P\omega \wedge \omega - \omega \wedge d_P\omega. \tag{85}$$

This results in

$$\begin{aligned} dF_i &= ds_i{}^*\Omega = s_i{}^* d_P\Omega = s_i{}^*(d_P\omega \wedge \omega - \omega \wedge d_P\omega) \\ &= ds_i{}^*\omega \wedge s_i{}^*\omega - s_i{}^*\omega \wedge ds_i{}^*\omega = dA_i \wedge A_i - A_i \wedge dA_i \\ &= F_i \wedge A_i - A_i \wedge F_i = -[A_i, F_i]. \end{aligned}$$

So we find the local identity

$$\mathcal{D}_i F_i = dF_i + [A_i, F_i] = 0, \tag{86}$$

where we defined the **covariant derivative**

$$\mathcal{D}_i = d + [A_i, \quad]. \tag{87}$$

3 The topology of principal bundles

We will now discuss some aspects of the topology of gauge bundles. As we have seen, pure Yang-Mills theory can be described using only principal bundles, so we restrict ourselves to a discussion of the topology of principal bundles. Again, this section is mainly influenced by [1], [4] and [3]. For more advanced treatments and different perspectives, we recommend [8], [9] and [7].

3.1 Aspects of homotopy theory

We start by making some simple remarks about the classification of topological spaces. Usually in topology, two spaces are considered equivalent if they can continuously be deformed into each other. In other words, they are considered topologically the same, if there exists a homeomorphism between them (for differentiable manifolds this would have to be diffeomorphism). Classifying spaces up to homeomorphism is a difficult thing to do. The idea is to find as much topological invariants (in general, numbers that do not depend on continuous parameters) of a type of space as possible. Finding a full set of invariants that completely classifies a space is rather difficult. The converse is however easily stated:

> If two topological spaces have different topological invariants, they are not homeomorphic, hence not topologically equivalent.

Since classification up to homeomorphism is such a difficult task, one can try to answer somewhat easier problems. For instance, one can try to classify spaces up to homotopy. Two spaces are said to be homotopic to each other if one can be mapped to the other in a continuous way, but this map need not have an inverse. For instance a circle and a cylinder are homotopic (one can continuously shrink the cylinder until its length disappears), but clearly not homeomorphic. In an intuitive sense, homotopic equivalence occurs when in the process of deforming one space to another, a part of the space is 'lost', so that it becomes impossible to define the reverse process.

For the moment, we are interested in homotopic equivalence classes of loops on a differentiable manifold M. It turns out that these reveal a very interesting group structure. We will only sketch this construction and state the results that we need for the remainder of these notes. By a (based) loop we will mean a map $\alpha : I = [0,1] \to M : t \mapsto \alpha(t)$, such that $\alpha(0) = \alpha(1) = x \in M$, the base point of the loop. It turns out that not these loops by themselves exhibit a group structure, but rather their equivalence classes under homopoty.

Definition 14. Two loops α and β based at the point $x \in M$ are **homotopic** to each other if there exists a continuous map

$$H : I \times I \to: M : (s,t) \mapsto H(s,t) \tag{88}$$

such that

$$\begin{array}{ll} H(s,0) = \alpha(s) \text{ and } H(s,1) = \beta(s), & \forall s \in I, \\ H(0,t) = H(1,t) = x, & \forall t \in I. \end{array}$$

$H(s,t)$ is called the **homotopy** between α and β.

One can show that this is an equivalence relation $\sim$ (reflexive, symmetric, transitive) between based loops. We will denote the equivalence class or homotopy class by $[\alpha]$. So we have that

$$\alpha \sim \beta \Rightarrow [\alpha] = [\beta]. \tag{89}$$

In other words, two loops are considered the same if one can continuously deform one into the other. One can now define the 'product' $\alpha \circ \beta$ of two loops as the result of first going through α and then through β. The inverse α^{-1} of a loop is then just going through the loop α in reverse order and the unit element is the constant loop $\alpha(t) = x$, $\forall t \in I$. It is clear that $\alpha \circ \alpha^{-1}$ are not equal, but homotopic to the identity. This is why only the homotopy classes of loops exhibit a group structure.

Definition 15. The group formed by the homotopy classes of loops based at x on a manifold M is called the **fundamental group** or first homotopy group $\Pi_1(M,x)$.

One can show that, if the manifold is arc-wise connected, the fundamental groups at different points are isomorphic. In that case we just refer to the fundamental group of the manifold $\Pi_1(M)$. If two manifolds are homotopic (of the same homotopy type) one can show that their fundamental groups are isomorphic. Since homotopy is a weaker than homeomorphy, if the fundamental group is invariant under homotopy, it must certainly be under homeomorphy. So we arrive at the following conclusion.

Theorem 8. The fundamental group is invariant under homeomorphisms and hence is a topological invariant.

As an example, let us look at $\Pi_1(S^1) = \Pi_1(U(1))$. This basically means that we want to classify maps from one circle to another. Intuitively, we know that one circle can wind the other an integer n times. This is accomplished by functions of the form

$$g_{n,a} : I \to S^1 : t \mapsto g_{n,a}(t) = e^{i(nt+a)}, \quad a \in \mathbb{R}. \tag{90}$$

It's easily shown that two maps $g_{n,a}$ and $g_{m,b}$ are homotopic to each other for any a, $b \in \mathbb{R}$ if $n = m$, but that for $n \neq m$ they are homotopically distinct. This means that we have homotopy classes

$$[n] \equiv [g_n], \tag{91}$$

where $g_n \equiv g_{n,0} = e^{int}$ is a good representative for each equivalence class. In this easy case one calls the integer n the degree or winding number of the map and we find that homotopy classes are characterized by their winding number. This winding number can be represented by the integral

$$n = \frac{1}{2\pi i}\int_0^{2\pi} dt\, g_n(t)^{-1}\frac{d}{dt}g_n(t). \tag{92}$$

From this it is clear that if $f_1(t)$ represents a map of winding number one, a map of winding number n is represented by $f_n(t) = f_1(t)^n$. It is also clear that the product of $[g_n]$ and $[g_m]$ is nothing but $[g_{n+m}]$, which means that the fundamental group is isomorphic to the additive group $\mathbb{Z}$. So we get the well known result $\Pi_1(S^1) = \mathbb{Z}$. In general $\Pi_1(M)$ can be non-Abelian.

One can also define the higher homotopy groups by looking at homotopy classes of maps from higher dimensional spheres to a manifold. More concretely, one looks at maps $\alpha : I^n \to M$, such that the entire boundary of the n-dimensional cube, ∂I^n maps to a single point $x \in M$. Going through the same procedure as for $\Pi_1(M)$ one arrives at the higher homotopy groups $\Pi_n(M)$, which are always Abelian. In general these homotopy groups are quite hard to calculate. However, in the examples we will be studying, the maps we want to classify are always between two spaces of the same dimension. In that case, there is the notion of the (Brouwer) degree of a map, which is somewhat easier to handle. In the end, it will give us the same information as the related homotopy groups would provide.

Definition 16. Consider a map $\phi : M \to N$, where $\dim M = \dim N = n$ and let Ω be a normalized volume form on N. The **Brouwer degree** of this map is defined as

$$\deg(\phi) = \int_M \phi^*\Omega, \quad \int_N \Omega = 1 \tag{93}$$

This definition does not depend on the volume form chosen, since the difference of two normalized volume forms has to be exact (its integral over N has to vanish) and the pull-back commutes with the exterior derivative. In addition one can show that the degree is an integer and hence has to be a topological invariant. We will show below that the degree we defined in (92) for the circle can interpreted exactly in this way.

We would like to study the topology of the simplest non-trivial bundles. Since, as we already mentioned, a bundle over a contractible space is always trivial, bundles over $\mathbb{R}^n$ are always trivial. The next simplest thing to study are bundles over n-spheres S^n and since these bundles are also very relevant and interesting for physics, we will mainly focus on these. One can always cover an n-sphere by two patches, say the north and the south hemisphere. The intersection of these two patches is homotopic to the equator, an $(n-1)$-sphere. This means that to classify a principal G-bundle over S^n, one would have to classify the transition functions on S^{n-1}, that is all maps from S^{n-1} to G. As we already discussed, a very interesting object to study in this regard is the homotopy group $\Pi_{n-1}(G)$. Later on we will look at $U(1)$-bundles over S^2 (Dirac monopoles) and $SU(2)$-bundles over S^4 (instantons). This requires knowledge of the groups $\Pi_1(U(1))$ and $\Pi_3(SU(2))$, respectively. $\Pi_1(U(1)) = \mathbb{Z}$ was already considered in the previous example, so let's now focus on $\Pi_3(SU(2))$.

We will again use the fact that homotopy classes of maps from S^3 to S^3 are characterized by their topological degree. First of all, we have to find a well defined volume form on $SU(2)$. For a general compact (matrix) group manifold, this is done as follows. At every point $g \in G$, one can define the left invariant $\mathfrak{g}$-valued Maurer-Cartan form $\Theta = g^{-1}dg$, as discussed in subsection (2.1). From this one can define a well-defined bi-invariant (left- and right-invariant) volume form on G.

For instance, for $SU(2)$ a normalized volume form is given by

$$\Omega = \frac{1}{24\pi^2} \operatorname{Tr}\left(g^{-1}dg \wedge g^{-1}dg \wedge g^{-1}dg\right), \tag{94}$$

where,

$$g = c_0 1_2 + c_i \tau_i \,, \qquad c_0^2 + c_i c_i = 1, \tag{95}$$

and τ_i, $i \in \{1,2,3\}$ are the Pauli matrices. Let (with a slight abuse of notation) $g : S^3 \to SU(2) : x \mapsto g(x)$. Then the degree of this map, according to equation (93), is given by

$$\deg(g) = \frac{1}{24\pi^2} \int_{S^3} \operatorname{Tr}\left(g^{-1}dg \wedge g^{-1}dg \wedge g^{-1}dg\right), \tag{96}$$

where the integrand should now be interpreted as the pull-back $g^*\Omega$. In other words, dg should now be interpreted as $\partial_i g dx^i$. Again, equation (96) will always give an integer n and since this degree fully characterizes elements of $\Pi_3(SU(2))$, we find that $\Pi_3(SU(2)) = \mathbb{Z}$.

In the case $G = U(1)$, the Maurer-Cartan form itself is a bi-invariant volume form, so we can take

$$\Omega = \frac{1}{2\pi i}\, g^{-1}dg \,, \qquad g \in U(1) \tag{97}$$

For the map $g_n : S^1 \to U(1) : t \mapsto g_n(t)$ we considered above we get

$$g_n{}^*\Omega = \frac{1}{2\pi i}\, g_n(t)^{-1} \frac{dg_n(t)}{dt}\, dt \,, \tag{98}$$

so that the integer n we defined for the circle above, can rightfully be called the degree of the map g_n, $\deg(g_n) = n$.

3.2 Characteristic classes

Besides Homotopy there are of course many different ways to construct topological invariants. An important example are groups generated by (co)homology classes of a manifold. We will now focus on certain integer cohomology classes constructed from polynomials in the field strength of a bundle, called characteristic classes. First we define invariant polynomials.

Definition 17. Let $\mathfrak{g}$ be the Lie algebra of some G. A totally symmetric and n-linear polynomial

$$P(X_1, ..., X_i, ..., X_j, ..., X_n) = P(X_1, ..., X_j, ..., X_i, ..., X_n), \quad \forall\, i, j, \tag{99}$$

where X_i, $i \in \{1, ..., n\}$ are elements of $\mathfrak{g}$, is called a **symmetric invariant** (or characteristic) **polynomial** if

$$P(g^{-1}X_1 g, ..., g^{-1}X_n g) = P(X_1, ..., X_n), \quad g \in G. \tag{100}$$

An immediate consequence of this definition is that (take g close to the identity, $g = 1 + tY$, and expand (100) to first order in t),

$$\sum_{i=1}^{n} P(X_1, ..., X_{i-1}, [Y, X_i], X_{i+1}, ..., X_n) = 0. \tag{101}$$

This will be of use to us later.

Definition 18. An **invariant polynomial** of degree n is defined as a symmetric invariant polynomial with all its entries equal,

$$P_n(X) \equiv P(\underbrace{X, ..., X}_{n}) \equiv P(X^n) \tag{102}$$

Now we want to extend this definition to $\mathfrak{g}$-valued differential forms on a manifold. A $\mathfrak{g}$-valued p-form, α_i, we write as (no sum over i) $\alpha_i = \eta_i X_i$, where η_i is an ordinary p-form and X_i again an element of $\mathfrak{g}$. We then extend the previous definition as follows:

Definition 19. An invariant polynomial for $\mathfrak{g}$-valued forms is defined as

$$P(\alpha_1, ..., \alpha_n) = P(X_1, ..., X_n)\, \eta_1 \wedge ... \wedge \eta_n. \tag{103}$$

The diagonal combination is again called an invariant polynomial of degree n,

$$P_n(\alpha) = P(\alpha^n) = P(X^n)\, \eta \wedge ... \wedge \eta. \tag{104}$$

Let β be a $\mathfrak{g}$-valued 1-form. From (101) we find that

$$\sum_{i=1}^{n} (-)^{p_1+...+p_{i-1}} P(\alpha_1, ..., \alpha_{i-1}, [\beta, \alpha_i], \alpha_{i+1}, ..., \alpha_n) = 0, \tag{105}$$

where p_i is the degree of α_i and the minus signs arise from pulling the 1-form to the front each time. Equally easy to compute is

$$dP(\alpha_1, ..., \alpha_n) = \sum_{i=1}^{n} (-)^{p_1+...+p_{i-1}} P(\alpha_1, ..., \alpha_{i-1}, d\alpha_i, \alpha_{i+1}, ..., \alpha_n). \tag{106}$$

Adding both equations for the specific case $\beta = A$, where A is the local gauge potential associated to a connection on the bundle (we drop the index referring to the specific local patch for convenience) and recalling the expression for the covariant derivative $\mathcal{D} = d + [A, \ \ ]$, we find the important expression

$$dP(\alpha_1, ..., \alpha_n) = \sum_{i=1}^{n} (-)^{p_1+...+p_{i-1}} P(\alpha_1, ..., \alpha_{i-1}, \mathcal{D}\alpha_i, \alpha_{i+1}, ..., \alpha_n). \tag{107}$$

The objects we want to study are invariant polynomials in the field strength 2-form F, $P_n(F)$, because these turn out to have very interesting properties. We will work with a connection on a principal bundle, although the following results equally hold for its associated vector bundle. We are now ready to prove the following important theorem:

Theorem 9. Let $P_n(F)$ be an invariant polynomial, then

(i) $P_n(F)$ is closed, $dP_n(F) = 0$.

(ii) Let F and F' be local curvature 2-forms corresponding to two different connections on the same bundle. Then the difference $P_n(F) - P_n(F')$ is exact.

Note that, since $P_n(F)$ is closed, it can always locally be written as the d of something (Poincaré's lemma). The important point about this theorem is that the difference $P_n(F) - P_n(F') = dQ$ is exact in a global sense (which is of course the meaning of exact), meaning that we have to prove that Q is globally defined! (a point which is usually ignored in physics textbooks)

Proof:

(i) The first part of the theorem follows immediately from (107), because of the Bianchi identity $\mathcal{D}F = 0$, see (86).

(ii) To prove the second part, consider two 1-form gauge potentials A and A', both referring to the same system of local trivializations, and their respective 2-form field strengths F and F'. We define the homotopic connection[9]

$$A_t = A + t\theta, \quad \theta = A' - A, \tag{108}$$

so that $A_0 = A$ and $A_1 = A'$, and its field strength

$$\begin{aligned} F_t &= dA_t + A_t \wedge A_t = F + t(d\theta + A \wedge \theta + \theta \wedge A) + t^2 \theta \wedge \theta \\ &= F + t\mathcal{D}\theta + t^2 \theta \wedge \theta, \end{aligned} \tag{109}$$

where $\mathcal{D} = d + [A, \;\;]$ (note the sign convention in the definition of the commutator). We now differentiate F_t with respect to t,

$$\frac{d}{dt}F_t = \mathcal{D}\theta + 2t\theta \wedge \theta = d\theta + A_t \wedge \theta + \theta \wedge A_t = \mathcal{D}_t\theta \tag{110}$$

with the obvious notation $\mathcal{D}_t = d + [A_t, \;\;]$. Considering then the invariant polynomial $P_n(F_t)$, we get

$$\frac{d}{dt}P_n(F_t) = nP(\frac{d}{dt}F_t, \underbrace{F_t, ..., F_t}_{n-1}) = nP(\mathcal{D}_t\theta, F_t^{n-1}). \tag{111}$$

From equation (107) and $\mathcal{D}_t F_t = 0$, we know that $dP(\theta, F_t^{n-1}) = P(\mathcal{D}_t\theta, F_t^{n-1})$, so that we find that

$$\frac{d}{dt}P_n(F_t) = ndP(\theta, F_t^{n-1}). \tag{112}$$

Integrating this from $t = 0$ to $t = 1$, we find

$$P_n(F') - P_n(F) = dQ_{2n-1}(A', A), \tag{113}$$

where we defined the **transgression** $Q_{2n-1}(A', A)$ as

$$Q_{2n-1}(A', A) = n \int_0^1 dt\, P(A' - A, F_t^{n-1}). \tag{114}$$

Note that $Q_{2n-1}(A', A)$ is indeed a gauge invariant and hence globally defined object, since under a gauge transformation (the inhomogeneous term cancels) $\theta' = g^{-1}\theta g$ and P is invariant.

□

Equation (113) is called a transgression formula and is quite important in the study of anomalies. We use it here to define the **Chern-Simons form**. Say that one can define a trivial connection $A' = 0$ on a bundle. This means that either the bundle is trivial or that we are working on a local patch. We know that since $P_n(F)$ is closed, it is locally exact; it can locally be written as the d of a Chern-Simons form. The transgression formula provides a means for calculating this Chern-Simons form. Indeed, from (113) we find

$$P_n(F) = dQ_{2n-1}(A), \tag{115}$$

[9] We are a bit sloppy here, because we should define this homotopy locally on a patch, where it is clear that this can be done. However, since both connections are defined on the same bundle (same set of transition functions) this turns out to be possible globally.

where we defined the Chern-Simons form

$$Q_{2n-1}(A) \equiv Q_{2n-1}(A,0) = n\int_0^1 dt\, P(A, F_t^{n-1}), \tag{116}$$

and now,

$$A_t = tA, \quad F_t = t\,dA + t^2 A\wedge A = tF + (t^2-t)A\wedge A \tag{117}$$

We see that, given an invariant polynomial $P_n(F)$, we can always construct the associated Chern-Simons form $Q_{2n-1}(A)$ from (116). Note that $P_n(F)$ is a $2n$-form, while Q_{2n-1} is a $(2n-1)$-form.

Since an invariant polynomial in F, $P_n(F)$, is closed and generically non-trivial, it represents a non-trivial (de Rham) cohomology class $[P_n(F)] \in H^{2n}(M,\mathbb{R})$, which is called a **characteristic class**. Since we have shown that the difference of two invariant polynomials defined with respect to two different connections is exact, we have by Stoke's theorem and for a manifold M without boundary, $\partial M = 0$,

$$\int_M P_n(F') - \int_M P_n(F) = \int_M dQ_{2n-1}(A',A) = \int_{\partial M} Q_{2n-1}(A',A) = 0. \tag{118}$$

This means that the integrals or periods $([P_n(F)], M)$ of these classes, usually called characteristic numbers, do not depend on the connection chosen, in other words, they are characteristic of the bundle itself (transition functions)! This makes characteristic classes very interesting objects to study the topology of fibre bundles. In contrast, Chern-Simons forms obviously do depend on the connection chosen, they are not even gauge invariant, but will prove to be very useful nonetheless. We now go on to define some examples of characteristic classes and Chern-Simons forms which are useful in the study of gauge theories.

3.3 Chern classes and Chern characters

Let P be a principal bundle, with structure group $G = GL(k,\mathbb{C})$ or a subgroup thereof ($U(k)$, SU(k),...)[10]. The **total Chern class** is defined by (the normalization of F is for later convenience)

$$c(F) = \det\left(1 + \frac{i}{2\pi}F\right). \tag{119}$$

Since F is a 2-form, $c(F)$ is a sum of forms of even degrees,

$$c(F) = 1 + c_1(F) + c_2(F) + ... \tag{120}$$

where $c_n(F) \in \Lambda^{2n}M$ is called the n-th **Chern class**[11]. If $\dim M = m$, all Chern classes, $c_n(F)$ of degree $2n > m$, vanish. In general it can be quite cumbersome to compute this determinant for higher dimensional manifolds. Therefore we will diagonalize the matrix $\frac{i}{2\pi}F$ (if for instance $G = SU(k)$, F is anti-hermitian, so iF is hermitian and can be diagonalized by an $SU(k)$ rotation g) to a matrix $\widetilde{F}$, with 2-forms x_i on the diagonal. This leads to

$$\begin{aligned}
\det(1+\widetilde{F}) &= \det[\mathrm{diag}(1+x_1,...,1+x_k)] = \prod_{i=1}^{k}(1+x_i) \\
&= 1 + (x_1 + ... + x_k) + (x_1x_2 + ... + x_{k-1}x_k) + ... + (x_1x_2...x_k) \\
&= 1 + \mathrm{Tr}\,\widetilde{F} + \frac{1}{2}\left[(\mathrm{Tr}\,\widetilde{F})^2 - \mathrm{Tr}\,\widetilde{F}^2\right] + ... + \det\widetilde{F}.
\end{aligned} \tag{121}$$

[10] One can equally well take the bundle to be an associated complex vector bundle E with one of the mentioned structure groups.

[11] Strictly speaking, this is a representative of the n-th Chern class, but we will follow the rest of the world in calling these Chern classes by themselves.

Note that in the second line we encounter the elementary symmetric functions of $\{x_i\}$ and that all manipulations are well defined since the x_i are 2-forms and thus commute (the wedge product is always understood). For an invariant polynomial $P_n(F) = P_n(g^{-1}Fg) = P_n(2\pi\widetilde{F}/i)$ (note that the trace always guaranties invariance), so we find the following expressions for the Chern classes:

$$\begin{aligned} c_1(F) &= \operatorname{Tr}\widetilde{F} = \frac{i}{2\pi}\operatorname{Tr}F && (122)\\ c_2(F) &= \frac{1}{2}\left[(\operatorname{Tr}\widetilde{F})^2 - \operatorname{Tr}\widetilde{F}^2\right] = \frac{1}{8\pi^2}\left[\operatorname{Tr}(F\wedge F) - \operatorname{Tr}F\wedge\operatorname{Tr}F\right] && (123)\\ &\vdots \\ c_k(F) &= \det\widetilde{F} = \left(\frac{i}{2\pi}\right)^k \det F && (124)\end{aligned}$$

To show how the computation of Chern-Simons forms goes about, let's start with a ridiculously easy example. Consider a $U(1)$-bundle over some 2-dimensional manifold. The only Chern class which can be defined is $c_1(F)$ and obviously, since locally $F = dA$, we find

$$c_1(F) = d\left(\frac{i}{2\pi}A\right), \quad \text{so that} \quad Q_1(A) = \frac{i}{2\pi}A. \tag{125}$$

Killing a fly with a jackhammer, we now use formula (116) to compute the same thing

$$Q_1(A) = \int_0^1 dt\, P(A) = \int_0^1 dt\, c_1(A) = \int_0^1 dt\, \frac{i}{2\pi}A = \frac{i}{2\pi}A. \tag{126}$$

Now that we have earned some trust in (116), we compute something less trivial. Consider an $SU(2)$-bundle over a 4-dimensional manifold. Since for $SU(k)$ we have that $\operatorname{Tr}F = 0$, the first Chern class vanishes. Let's try to compute the Chern-Simons form related to the second Chern class,

$$\begin{aligned} Q_3(A) &= 2\int_0^1 dt\, P(A, F_t) = \frac{1}{4\pi^2}\int_0^1 dt\ \operatorname{Tr}(A\wedge F_t)\\ &= \frac{1}{4\pi^2}\int_0^1 dt\ \operatorname{Tr}(tA\wedge dA + t^2 A\wedge A\wedge A)\\ &= \frac{1}{8\pi^2}\operatorname{Tr}\left(A\wedge dA + \frac{2}{3}A\wedge A\wedge A\right). \end{aligned} \tag{127}$$

which is of course the most famous example of a Chern-Simons form in physics.

Since the periods of Chern classes are independent of the connection, these numbers, called **Chern numbers**, are denoted as

$$c_n \equiv ([c_n(F)], M) = \int_M c_n(F). \tag{128}$$

One can show that on a compact manifold, these numbers are always integers, $c_n = k$, a phenomenon called topological quantization. We will see instances of this where the Chern numbers compute the monopole charge or instanton number later.

We will now briefly discuss another characteristic class called the Chern character, because it has some properties which make it easier to compute than Chern classes (one can afterwards compute the Chern classes from the Chern characters) and because it appears in the Atiyah-Singer index theorem. The **total Chern character** (again for $G \subseteq GL(k,\mathbb{C})$) is defined by

$$ch(F) = \mathrm{Tr}\exp\left(\frac{i}{2\pi}F\right) = \sum_n \frac{1}{n!}\,\mathrm{Tr}\left(\frac{i}{2\pi}F\right)^n . \tag{129}$$

This is again a sum over even forms, the **Chern characters**

$$ch_n(F) = \frac{1}{n!}\,\mathrm{Tr}\left(\frac{i}{2\pi}F\right)^n \tag{130}$$

By again diagonalizing iF to a diagonal matrix $\widetilde{F}$ with eigenvalues $\{x_i\}$, using

$$\mathrm{Tr}\exp(\widetilde{F}) = \sum_{i=1}^{k}\exp(x_i) = \sum_{i=1}^{k}(1 + x_i + \frac{1}{2}x_i^2 + ...), \tag{131}$$

and expressing the result in terms of elementary symmetric functions of $\{x_i\}$, one can relate the Chern characters to the Chern classes. A few examples are,

$$ch_0(F) = k \tag{132}$$
$$ch_1(F) = c_1(F) \tag{133}$$
$$ch_2(F) = -c_2(F) + \frac{1}{2}c_1(F)\wedge c_1(F). \tag{134}$$

Since for a Dirac monopole we only need $c_1(F) = ch_1(F)$ and for $SU(2)$-instantons $c_1(F) = 0$, so that $ch_2(F) = -c_2(F)$, we will not really see the difference between the two. We will mostly refer to the Chern class, when speaking about either of the two.

4 Some applications

We will now apply the formalism we developed to two standard examples of the topology of gauge bundles, Dirac monopoles and instantons. We will not talk about (the more interesting) non-Abelian 't Hooft-Polyakov monopoles, because these do not appear in pure gauge theory, but require a Higgs field, which is a section of an associated vector bundle and are not as such 'pure' examples of the topology of gauge bundles. For more on these and other applications of bundles to physics, see [1] - [7].

4.1 Dirac monopoles

Consider a magnetic monopole in Maxwell theory (Abelian) at the origin of 3-dimensional Euclidean space, $\mathbb{R}^3$. If q is the magnetic charge of the monopole, we can take the magnetic charge distribution to be $\rho(x) = 4\pi q\delta(x)$. Let B^i be the components of the magnetic field (a vector field on $\mathbb{R}^3$). From Maxwell's equations, we know that $\partial_i B_i(x) = q\delta(x)$, which has the spherically symmetric solution

$$B = \frac{q}{r^2}\frac{\partial}{\partial r}. \tag{135}$$

This expression becomes infinite at the origin, so that strictly speaking it is only a physically relevant solution on $\mathbb{R}^3_0 = \mathbb{R}^3\backslash\{(0,0,0)\}$. There are no non-trivial bundles over $\mathbb{R}^3$, but for $\mathbb{R}^3_0$ there is the possibility that there exists a fibre bundle description of this kind of monopole. Since $\mathbb{R}^3_0$ is homotopic to a sphere and we are considering a pure $U(1)$-gauge theory, the proper bundle setting is that of a principal $U(1)$-bundle over S^2, $P(U(1), S^2)$.

We described two ways for classifying this bundle. One was the characterization of $\Pi_1(U(1))$ by the topological degree or winding number of the map from the equator S^1 of S^2 to the fibre $U(1)$. The other was by computing the first Chern class $c_1(F)$ of a $U(1)$-connection on the sphere S^2.

Let's look at the second approach. The sphere can be covered by two patches U_N and U_S, corresponding to the northern and southern hemisphere respectively, with $U_N \cap U_S = S^1$. On each patch we have a local 1-form gauge potential, A^N on U_N and A^S on U_S. Since we are dealing with an Abelian structure group, the 2-form field strength F is gauge invariant. This means that on the equator

$$F|_{S^1} = dA^N = dA^S. \tag{136}$$

The first Chern number is computed as follows:

$$c_1 = \int_{S^2} c_1(F) = \frac{i}{2\pi}\int_{S^2} F = \frac{i}{2\pi}\left(\int_{U_N} F + \int_{U_S} F\right). \tag{137}$$

The field strength is locally exact on both hemispheres, so by Stoke's theorem we find,

$$c_1 = \frac{i}{2\pi}\left(\int_{\partial U_N} A^N + \int_{\partial U_S} A^S\right) = \frac{i}{2\pi}\int_{S^1}\left(A^N - A^S\right). \tag{138}$$

On the equator both potential 1-forms are related by a transition function $g \in U(1)$,

$$A^S = A^N + g^{-1}dg. \tag{139}$$

This leads to

$$c_1 = \frac{1}{2\pi i}\int_{S^1} g^{-1}dg, \tag{140}$$

which we recognize as the winding number of the map $g : S^1 \to U(1)$. We see that for a map g_n of winding number n, we find that $c_1 = \deg(g_n) = n$.

To connect to physics, we now write down an explicit solution. The 1-form β associated to the vector field B in (135) is (we use spherical coordinates with metric components $\eta_{rr} = 1$, $\eta_{\theta\theta} = r$ and $\eta_{\varphi\varphi} = r\sin\theta$),

$$\beta = \frac{q}{r^2}dr. \tag{141}$$

The field strength 2-form F is the Hodge dual to this (to find complete agreement with the general theory, we need to include the Lie algebra factor i),

$$F = i * \beta = i\sqrt{\det\eta}\, B_r\, \varepsilon^r{}_{\theta\varphi}\, d\theta \wedge d\varphi = iq\sin\theta\, d\theta\wedge d\varphi. \tag{142}$$

This is the field strength 2-form which represents a Dirac monopole of charge q. A possible gauge potential 1-form that leads to this field strength is $A = -i\cos\theta\, d\varphi$. Since spherical coordinates are badly behaved along the entire z-axis ($\theta = 0, \pi$), we can't use this potential for either hemisphere. We can however define,

$$\begin{aligned} A^N &= iq(1-\cos\theta)d\varphi \quad \text{on} \quad U_N && (143)\\ A^S &= -iq(1+\cos\theta)d\varphi \quad \text{on} \quad U_S, && (144)\end{aligned}$$

which are well defined on their respective patches and lead to the same F. On the equator, we find

$$A^N = A^S + 2iqd\varphi. \tag{145}$$

This completely agrees with equation (139) for

$$g_n = e^{in\varphi}, \quad n = 2q, \tag{146}$$

so that a monopole of charge q corresponds to a $U(1)$-bundle over S^2 with winding number $n = 2q$, or, to put it differently, corresponds to an element $[2q]$ of $\Pi_1(U(1))$.

We conclude this subsection with an important note. The above reasoning might falsely cause one to believe that one does not need quantum mechanics to prove the quantization of magnetic charge. The main assumption we used, however, is that one can use a bundle to describe the magnetic field on a sphere in the first place. This manifests itself in a gauge transformation by $g^{-1}dg$ in eq. (139), instead of by just a general closed one-form on the equator. This is equivalent to the assumption that the magnetic field is described by an integral 2-form (a first Chern class) as opposed to a generic 2-form, which is not integral. The integrality of the magnetic field of a monopole is usually proved by considering the wave function of a quantum mechanical particle in the neighborhood of the monopole. To summarize: classically, the magnetic field is just a (non-integral) 2-form (which consequently is not the curvature of a bundle), while quantum mechanically, it is an integral 2-form, which means that it can be seen as the curvature (first Chern class) of a bundle.

4.2 Holonomy and the Aharonov-Bohm effect

We already briefly discussed holonomy in subsection 2.2, but let us come back to it in little more detail. Consider a principal bundle $P(M, G)$ with a connection 1-form ω. Let γ be a curve on M and γ_P be a horizontal lift. Suppose for the moment that γ is contained within a single patch U and let $s : U \to P$ be a section on U. This means that $\gamma_P(t) = s(\gamma(t))g(t)$. The aim is to compute $g(t)$ to have a local description of parallel transport (local because this description depends on s).

If $X \in T\gamma M$ is the tangent to γ, the tangent to γ_P is $X_P = \gamma_{P*}X \in T_{\gamma_P}P$. Since γ_P is horizonal, we have that $X_P \in H_{\gamma_P}P$ or $\omega(X_P) = 0$. According to (64), this means

$$g^{-1}\pi^* A(X_P)g + g^{-1}d_P g(X_P) = 0. \tag{147}$$

Using that $\pi_* X_P = \pi_*\gamma_{P*}X = X$, we find on M,

$$g^{-1}A(X)g + g^{-1}dg(X) = 0, \tag{148}$$

or

$$dg(X) = X(g) = -A(X)g. \tag{149}$$

Since X is tangent to γ, we have

$$X(g) = \frac{d}{dt}g(\gamma(t)) \quad \text{and} \quad A(X) = A_a X^a = A_a \frac{d}{dt}x^a(\gamma(t)), \tag{150}$$

where $A = A_a dx^a$. Writing $g(\gamma(t)) = g(t)$ and $x^a(\gamma(t)) = x^a(t)$, this leads to

$$\frac{dg(t)}{dt}g(t)^{-1} = -A_a \frac{dx^a(t)}{dt}. \tag{151}$$

For $G = U(1)$ this has the solution (suppose that $g(0) = e$)

$$g(t) = \exp\left[-\int_0^t dt\, A_a \frac{dx^a(t)}{dt}\right] = \exp\left[-\int_{\gamma(0)}^{\gamma(t)} A_a dx^a\right] \tag{152}$$

For short, we can write

$$g(t) = \exp\left[-\int_\gamma A\right]. \tag{153}$$

For a non-Abelian structure (gauge) group, this is modified to

$$g(t) = P\exp\left[-\int_\gamma A\right]. \tag{154}$$

where P indicates that the exponential is defined by its power series expansion and that the matrix-valued forms should always be path ordered. In physics this is called a Wilson line.

If $s' = sh$ is another section on U (or on an overlap with another patch U'), related to s by a group element $h \in G$, one can show that if $\gamma_P(t) = s'(\gamma(t))g'(t)$, we find

$$g'(t) = h^{-1}(t)g(t)h(0). \tag{155}$$

This shows that if $G = U(1)$ and if $\gamma(0) = \gamma(1)$, so that $h(1) \equiv h(\gamma(1)) = h(\gamma(0)) = h(0)$, then

$$g_\gamma \equiv \exp\left[-\oint_\gamma A\right], \tag{156}$$

called a Wilson loop, is gauge invariant. g_γ is nothing but an element of the holonomy group $Hol(P)$ we discussed in subsection 2.2. We see that in the non-Abelian case this procedure doesn't lead to a gauge invariant quantity, but

$$g'_\gamma = h^{-1}g_\gamma h, \quad h = h(0) = h(1). \tag{157}$$

If we take the trace of this though, we do get a gauge invariant quantity. In non-Abelian gauge theories the Wilson loop is thus defined as the trace of the Wilson line around a closed loop,

$$W_\gamma = \mathrm{Tr}\, g_\gamma = \mathrm{Tr}\, P\exp\left[-\oint_\gamma A\right], \tag{158}$$

and equals the trace of the holonomy at $p = \gamma_P(0)$.

To illustrate this, consider a solenoid along the x_3-axis in $\mathbb{R}^3$. The $U(1)$ magnetic field is uniform in the interior of the solenoid and practically vanishing outside of it. In the limit of infinitely thin and long solenoid, the magnetic field is strictly 0 in the exterior region, which is $\mathbb{R}^3_0$, but there still is a non-zero flux Φ associated to it. By Stoke's theorem, this means that the gauge potential A cannot be zero in the exterior region, since for any curve γ encircling the solenoid, which spans a surface A_γ

$$\oint_\gamma A = \int_{A_\gamma} dA = \int_{A_\gamma} F = \Phi. \tag{159}$$

Since $F = 0$ outside of the solenoid, there is no classical effect on a particle moving alongside it. Quantum mechanically however, it is known that A can have a physical meaning. To see this, consider a wave function ψ of a particle moving in the x_1x_2-plane perpendicular to the solenoid. This is described by a section of a complex line bundle over $\mathbb{R}^2_0$ associated to the principal bundle $P(\mathbb{R}^2_0, U(1))$ by the obvious representation

$$\psi \to \rho(e^\alpha)\psi = e^\alpha\psi. \tag{160}$$

In a path integral approach, every path γ is weighed by a factor

$$g(t) = e^{iS_\gamma}, \quad S_\gamma = \int_\gamma dt\, L, \tag{161}$$

where the important part of the Lagrangian L is the part involving the gauge potential,

$$L = A_i(x) \frac{dx^i}{dt}. \tag{162}$$

In other words, every path is weighed by a factor (note that we absorbed the Lie algebra factor into A to make contact with our formalism),

$$g(t) = \exp \int_\gamma A. \tag{163}$$

In a double slit experiment, with the solenoid placed between the slits, part of the wave function ψ_1 will move along path γ_1 above the solenoid and part ψ_2 will move along a path γ_2 underneath it. The total wave function is[12]

$$\begin{aligned} \psi &= \exp\left[\int_{\gamma_1} A\right]\psi_1 + \exp\left[\int_{\gamma_2} A\right]\psi_2 \\ &= \exp\left[\int_{\gamma_1} A\right]\left\{\psi_1 + \exp\left[\int_{\gamma_2} A - \int_{\gamma_1} A\right]\psi_2\right\}. \end{aligned} \tag{164}$$

What is important of course, is the phase difference,

$$\int_{\gamma_2} A - \int_{\gamma_1} A = \oint_\gamma A, \quad \gamma = \gamma_2 - \gamma_1. \tag{165}$$

We see that the probability to find a particle at a certain point on the screen is influenced by the gauge potential through the holonomy of the connection

$$|\psi|^2 = |\psi_1 + g_\gamma \psi_2|^2 = |\psi_1 + e^\Phi \psi_2|^2. \tag{166}$$

Note that the only reason why $\oint A$ can have physical meaning is because it is gauge invariant (independent of local trivializations).

G-bundles over a circle are classified by $\Pi_0(G)$, implying that a $U(1)$-bundle over S^1 is necessarily trivial. Hence, this example shows that a trivial bundle can contain non-trivial physics, i.e. the Aharonov-Bohm effect.

4.3 Instantons

Instantons are traditionally defined as smooth finite action solutions of Yang-Mills theory on 4-dimensional Euclidian space $\mathbb{R}^4$. We will only consider the case of $SU(2)$. There exist no non-trivial bundles over $\mathbb{R}^4$, but the finiteness of the action imposes boundary conditions at infinity, which allow for the existence of topologically non-trivial solutions of the field equations. This is seen as follows: To get a finite action, the field strength has to go to zero (fast enough) at infinity. This means that along a sufficiently large 3-sphere, S^3_∞, the gauge potential has to be pure gauge

$$A|_{S^3_\infty} = g^{-1} dg \implies F|_{S^3_\infty} = 0, \tag{167}$$

[12] We are mixing path integral and ordinary QM arguments. Since the only real importance is wether the path passes underneath or above the solenoid, the path integral essentially reduces to the sum of 2 paths, so that both viewpoints essentially give the same information.

so these solutions are classified by maps $g : S^3_\infty \to SU(2)$. The reason for their stability is the fact that one cannot change the homotopy class of this map while keeping the total action finite. Computing the second Chern number, using that $c_2(F) = dQ_3(A)$ (and assuming that there is no contribution outside of S^3_∞),

$$\begin{aligned}
c_2 &= \int_{\mathbb{R}^4} c_2(F) = \int_{S^3_\infty} Q_3(A) = \frac{1}{8\pi^2}\int_{S^3_\infty} \mathrm{Tr}\left(A\wedge dA + \frac{2}{3}A\wedge A\wedge A\right) \\
&= \frac{1}{8\pi^2}\int_{S^3_\infty} \mathrm{Tr}\left(F\wedge A - \frac{1}{3}A\wedge A\wedge A\right) = -\frac{1}{24\pi^2}\int_{S^3_\infty}\mathrm{Tr}\left(A\wedge A\wedge A\right) \\
&= -\frac{1}{24\pi^2}\int_{S^3_\infty}\mathrm{Tr}\left(g^{-1}dg\wedge g^{-1}dg\wedge g^{-1}dg\right). \qquad (168)
\end{aligned}$$

Which is (up to a sign) exactly the topological degree of the map $g : S^3_\infty \to SU(2)$.

To gain more control over the situation and allow for a bundle description of instantons, we consider a one-point compactification of $\mathbb{R}^4$ to S^4, by adding to it the point at infinity, $\mathbb{R}^4 \cup \{\infty\} = S^4$. This means that we want to look at principal $SU(2)$-bundles over S^4, $P(SU(2), S^4)$. As we saw in subsection 3.1, this is classified by $\Pi_3(SU(2))$, while on the other hand one can compute the second Chern number of an $SU(2)$-connection on S^4.

Again, we can cover the 4-sphere by two open sets, the northern and southern hemisphere, U_N and U_S respectively. This time, the field strength 2-form is not invariant, but

$$F^N = dA^N + A^N\wedge A^N, \quad F^S = dA^S + A^S\wedge A^S, \qquad (169)$$

with

$$A^N = g^{-1}A^S g + g^{-1}dg \Rightarrow F^N = g^{-1}F^S g. \qquad (170)$$

Let's try to compute the second Chern number

$$\begin{aligned}
c_2 &= \int_{S^4} c_2(F) = \int_{U_N} dQ_3(A^N) + \int_{U_S} dQ_3(A^S) = \int_{S^3}\left(Q_3(A^N) - Q_3(A^S)\right) \\
&= \frac{1}{8\pi^2}\int_{S^3}\mathrm{Tr}\left(F^N\wedge A^N - \frac{1}{3}A^N\wedge A^N\wedge A^N - F^S\wedge A^S + \frac{1}{3}A^S\wedge A^S\wedge A^S\right) \\
&= -\frac{1}{8\pi^2}\int_{S^3}\mathrm{Tr}\left(\frac{1}{3}g^{-1}dg\wedge g^{-1}dg\wedge g^{-1}dg - d(g^{-1}A^N\wedge dg)\right) \qquad (171)
\end{aligned}$$

where in the last step we used the gauge transformations (170), the identity

$$dg^{-1} = -g^{-1}dgg^{-1}, \qquad (172)$$

and the fact that for three $\mathfrak{g}$-valued 1-forms

$$\mathrm{Tr}(\alpha\wedge\beta\wedge\gamma) = \mathrm{Tr}(\gamma\wedge\alpha\wedge\beta). \qquad (173)$$

Since S^3 has no boundary, we arrive at the expected result

$$c_2 = \frac{1}{24\pi^2}\int_{S^3}\mathrm{Tr}\left(g^{-1}dg\wedge g^{-1}dg\wedge g^{-1}dg\right). \qquad (174)$$

Again we find that $c_2 = \deg(g) = k$, as described in (96). We conclude that the classification by the second Chern class $[c_2(F)]$ is equivalent to the one given by $\Pi_3(SU(2))$. Both are characterized by the degree of the transition function of the bundle.

What does this have to do with instantons? As we already stated, we have to find finite action solutions of the action for $F = F_{ij} dx^i \wedge dx^j$, $F_{ij} = F_{ij}^{\ a} T_a$

$$S_E = \frac{1}{4} \int dx^4 \, F_{ij}^{\ a} F^{ij}_{\ a} = -\frac{1}{2} \int dx^4 \ \mathrm{Tr}\left(F_{ij} F^{ij}\right) = - \int \mathrm{Tr}(F \wedge *F), \qquad (175)$$

where $*F$ is the Hodge dual of F, in flat space

$$* F_{ij} = \frac{1}{2} \varepsilon_{ijkl} F^{kl}. \qquad (176)$$

For $\mathfrak{su}(2)$, the Lie algebra of $SU(2)$, we take the generators (in the defining representation) to be,

$$T_a = \frac{1}{2i} \, \tau_a, \qquad (177)$$

where τ_a, $a \in \{1,2,3\}$, are the Pauli matrices. The generators have the following properties

$$\mathrm{Tr}(\tau_a \tau_b) = 2\delta_{ab} \quad \Rightarrow \quad \mathrm{Tr}(T_a T_b) = -\frac{1}{2} \, \delta_{ab}, \qquad (178)$$

$$[\tau_a, \tau_b] = 2i \, \varepsilon_{ab}^{\ \ c} \, \tau_c \quad \Rightarrow \quad [T_a, T_b] = \varepsilon_{ab}^{\ \ c} \, T_c. \qquad (179)$$

To write the action in a convenient way, we calculate the following positive definite object

$$\begin{aligned} \frac{1}{4} \int (F_{ij}^{\ a} \pm *F_{ij}^{\ a})(F^{ij}_{\ a} \pm *F^{ij}_{\ a}) &= - \int \mathrm{Tr}(F \pm *F) \wedge *(F \pm *F) \\ &= -2 \int \mathrm{Tr}(F \wedge *F) \mp 2 \int \mathrm{Tr}(F \wedge F). \end{aligned} \qquad (180)$$

From this, we find

$$\begin{aligned} S_E &= -\frac{1}{2} \int \mathrm{Tr}(F \pm *F) \wedge *(F \pm *F) \pm \int \mathrm{Tr}(F \wedge F) \\ &\geq 8\pi^2 |k|. \end{aligned} \qquad (181)$$

Where we defined the **instanton number** k to be

$$k = -c_2 = ch_2. \qquad (182)$$

We see that the positive definite action S_E is bounded from below and that a minimum is attained when

$$F = \pm * F, \qquad (183)$$

called self-dual (SD) and anti-self-dual (ASD) instantons respectively. We have chosen the instanton number k in such a way that it is positive (negative) for (A)SD instantons, as one can see from (175). This shows that (181) does establish a lower bound. Since (A)SD instantons are minima of the action, they are solutions to the equations of motion.

4.4 Further applications and remarks

Characteristic classes are a very important ingredient for the Atiyah-Singer index theorem. It would take us too far to go into a lot of details, but we will try to sketch some aspects of this application of characteristic classes to the study of anomalies and moduli spaces. For more information on the use of fibre bundles in the study of anomalies, see [1] and [4]

The idea is to compute an analytic index of certain differential operators defined on a bundle by computing topological quantities of the bundle expressed as integrals of characteristic classes of the bundle. By an analytic index of an operator D, we mean

$$\mathrm{Ind}D = \dim\ \ker D - \dim\ \ker D^\dagger \tag{184}$$

where $D^\dagger$ is the adjoint of D with respect to an inner product,

$$(u, Dv) = (D^\dagger u, v). \tag{185}$$

For the analytic index to be well defined, we need that both ker D and ker $D^\dagger$ are finite dimensional. A (bounded) operator which satisfies these conditions is called a Fredholm operator. One can show that certain differential operators on compact boundaryless manifolds, called elliptic operators, are always Fredholm operators.

It is for these elliptic differential operators that Atiyah and Singer found another way to express the index, namely in terms of characteristic classes. One important example of where this is relevant in physics, is for the chiral anomaly. Consider a massless Dirac spinor coupled to an $SU(2)$ gauge theory on a 4-dimensional manifold M. Classically, there is a global chiral symmetry

$$\psi' = e^{i\gamma_5\alpha}\psi, \quad \bar{\psi}' = \bar{\psi}e^{i\gamma_5\alpha}, \tag{186}$$

which leads to a conserved current

$$\partial_a j_5^a = 0. \tag{187}$$

Quantum mechanically this symmetry is broken, so that it is anomalous. One can show that the right hand side of (187) is no longer zero, but that its integral over M is given by the index of the Dirac operator

$$\mathrm{ind}(i\nabla\!\!\!/_+) = \dim\ \ker(i\nabla\!\!\!/_+) - \dim\ \ker(i\nabla\!\!\!/_-) = n_+ - n_- \tag{188}$$

where

$$i\nabla\!\!\!/_\pm = i\gamma^a\nabla_a P_+ = i\gamma^a(\partial_a + A_a)\frac{1}{2}(1 \pm \gamma_5), \tag{189}$$

and $n_\pm$ are the number of positive and negative chirality zero modes of the Dirac operator, respectively. It is clear that if the index of this operator is nonzero, the chiral symmetry is broken. For this example, the Atiyah-Singer index theorem states that (if all relevant characteristic classes of the tangent bundle TM are zero),

$$\mathrm{ind}(i\nabla\!\!\!/_+) = \int_M ch_2(F) = ch_2, \tag{190}$$

so the index is given by the second Chern character of P. Since for $SU(2)$ this is equal to the instanton number k, the statement becomes,

$$\int_M dx^4\, \partial_a j_5^a = \mathrm{ind}(i\nabla\!\!\!/_+) = n_+ - n_- = ch_2 = k. \tag{191}$$

We see that the index computes the anomaly of the theory and shows the obstruction for the classical symmetry to become a quantum symmetry. Moreover, (191) shows that the instanton background breaks the symmetry. On the other hand if there are no instantons, the chiral symmetry is a symmetry in the full quantum theory.

Because the current in this case carries no group index, this is also called the Abelian anomaly. A similar discussion can be given for a non-Abelian anomaly.

The index theorem can for instance also be used to compute the dimension of the moduli space of instantons in $SU(N)$. One can show that the number of parameters to describe a general instanton (for a given winding number k) is related to the number of zero modes of a kind of Dirac operator. It would take us too far to discuss this in detail, but an index slightly different from the above one computes the number of zero modes of this operator. Using this, one can show that the moduli space of $SU(N)$ instantons with winding number k is $4kN$-dimensional [10].

We have seen two examples of integral cohomology classes at work, $[c_1(F)]$ for the monopole and $[c_2(F)]$ for the instanton. Both are elements of $H^p(M, \mathbb{Z})$, where $\dim M = p$ ($p = 2$ for the monopole and $p = 4$ for the instanton). They both represent an obstruction for the principal bundle P to be trivial. In this sense it's clear that they both represent an obstruction for P to have a section, this can only happen if P is trivial. Note also that in both cases, $\mathbb{Z}$ turns out to be $\Pi_{p-1}(G)$, namely $\Pi_1(U(1)) = \Pi_3(SU(2)) = \mathbb{Z}$. These turn out to be special cases of a far more general result.

Let $P(M, G)$ be a principal bundle, and let M_p be the p-dimensional skeleton of M. Define s_p to be a section of the bundle P defined only over the skeleton, i.e. $s_p : M_p \to \pi^{-1}(M_p)$. Generically, any given s_p can be extended to a section s_{p+1} over M_{p+1}. However, if $P(M, G)$ is not trivial, there will be obstructions to continuing such a chain of extensions. Without intending to be fully rigorous, the general result found in [8] can be stated as follows:

> Let p be the first dimension such that sections over M_{p-1} cannot be extended to sections over M_p. Then, the obstruction to building a section of P over an p-dimensional skeleton of M lies in the cohomology group $H^p(M, \Pi_{p-1}(G))$.

We will not try to delve much deeper into these very interesting matters, but note that $\Pi_{p-1}(G)$ need not be $\mathbb{Z}$. So the relevant cohomology classes need not be integer valued, but might be, for instance, $\mathbb{Z}_2$ valued, like Stiefel-Whitney classes. For instance, the non-triviality of the Möbius strip is measured by the first Stiefel-Whitney class, which belongs to the cohomology group

$$H^1(S^1, \Pi_0(\mathbb{Z}_2)) = H^1(S^1, \mathbb{Z}_2) = \mathbb{Z}_2. \tag{192}$$

This tells us that there are two ways to make a $\mathbb{Z}_2$ bundle over S^1.

The two physics examples we studied can be interpreted as follows: For the monopole, we had

$$H^2(S^2, \Pi_1(U(1))) = H^2(S^2, \mathbb{Z}) = \mathbb{Z}. \tag{193}$$

This means that, if the class is not the trivial element in cohomology, there is an obstruction to finding a section over a 2-dimensional skeleton of S^2. In this case, because this is incidentally the top class of the manifold, this means indeed that one cannot construct a section of $P(S^2, U(1))$, so that the bundle is non-trivial. A charge n monopole corresponds to a connection on a bundle which is in the class $[n]$ of $H^2(S^2, \Pi_1(U(1)))$, where n is the degree of the appropriate map or is the first Chern number c_1 (see subsection 4.1).

For instantons, the story is analogous. In this case the relevant object is,

$$H^4(S^4, \Pi_3(SU(2))) = H^4(S^4, \mathbb{Z}) = \mathbb{Z}. \tag{194}$$

This again describes the possible obstruction to finding a section of $P(S^4, SU(2))$, because this is again the top class. The instanton with instanton number k corresponds to a connection on a bundle in class $[k]$, where $k = -c_2 = ch_2$ or is again the degree of the appropriate map (see subsection 4.3). Note that for both the monopole and instanton, the gauge potential describes one possible connection on the bundle. There are many possible connection on a bundle in a certain class $[n]$, but they will all compute the same Chern class and be characterized by the same topological degree. In this sense, the bundle itself is more fundamental than the specific solution of the gauge theory we construct. Of course, this does not take away the meaning of these specific solutions. For instance, an instanton is still a minimum of the action, while other gauge configurations in the same topological class are generically not.

Notice that a characteristic class can only be viewed as an element of the group $H^p(M, \Pi_{p-1}(G))$ if it represents the *first* obstruction to the extendibility of sections over skeletons. Higher obstructions require a different interpretation. For example, if $H^2(M, \mathbb{Z})$ is not trivial for some four-dimensional M, then it is possible to have a non-trivial $U(1)$ bundle over M with non-zero first Chern class F. Although the second Chern class of a rank one bundle is always trivial, it is possible in this case to have a non-trivial second Chern character. This means that one can have a $U(1)$ instanton on a four-dimensional manifold even though $\Pi_3(U(1)) = 0$. This is possible because the second Chern character does not represent the first obstruction of the $U(1)$ bundle, and hence does not lie in $H^4(M, \Pi_3(U(1))$. Instead, in this case it lies in $H^4(M, \Pi_1(U(1))$. Although such instantons are not wide-spread objects in quantum field theory, they certainly make their appearance in string theory. For those who are familiar with string theory, the example in question is a D4 brane wrapped on a manifold with $H^2 \neq 0$, that carries lower-dimensional D2 and D0 charge. The D2 brane can be viewed as vortex or string living on the D4, whose charge is given by the first Chern class of the $U(1)$ bundle on the D4. The D0 brane can be viewed as a $U(1)$ instanton on the D4 (if one ignores the time direction), such that its charge is given by the second Chern character of the $U(1)$ bundle.

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A Some differential geometry

In this appendix we will quickly review some of the concepts of the theory of differentiable manifolds that we will use in the rest of these notes. We will also establish a lot of notation used throughout the main text.

A.1 Manifolds and tangent spaces

First of all, a differentiable manifold is roughly a smooth topological space, which locally looks like $\mathbb{R}^n$. By identifying an open subset of a manifold with an open subset of $\mathbb{R}^n$, the notion of differentiability of a function from $\mathbb{R}^n$ to $\mathbb{R}^m$ is passed on to one of a function from one manifold to another. This means that one can compare manifolds as smooth spaces. More importantly for these lectures, it allows for doing physics on them, much like on $\mathbb{R}^n$. Let's be a bit more concrete.

Definition 20. We call M a differentiable manifold if the following conditions are satisfied

(i) M is a topological space

(ii) M is equipped with a set of pairs $\{(U_i, \varphi_i)\}$, where $\{U_i\}$ is an open cover of M (all U_i are open sets and $M = \bigcup_i U_i$) and φ_i is a homeomorphism from U_i to an open subset of $\mathbb{R}^n$. n is called the dimension of M

(iii) On an overlap $U_i \cap U_j \neq \emptyset$, the map $\varphi_j \circ \varphi_i^{-1} : \mathbb{R}^n \to \mathbb{R}^n$ is continuously differentiable.

Again, given a local chart (U_i, φ_i) one can define physical quantities on U_i much like one would do on $\mathbb{R}^n$, where the φ_i define coordinates on U_i. The different patches U_i can however be glued together in a nontrivial way by the transition functions $\varphi_j \, \varphi_i^{-1}$, so that globally a manifold is a generalization of $\mathbb{R}^n$. We give this definition for completeness, but to keep the notation tractable we will be a bit sloppy throughout the text and keep the φ_i implicit. For instance, to define the derivative of a function $f : U_i \to \mathbb{R}$ at a point $x \in U_i$, one would have to consider instead $f \circ \varphi_i^{-1} : \mathbb{R}^n \to \mathbb{R}$ and use the definition of a derivative of a functional on $\mathbb{R}^n$. We will omit φ from this expression and treat f as though it were a function on $\mathbb{R}^n$.

First of all, we define the tangent space to $x \in M$.

Definition 21. Let $\gamma : [0, 1] \to U_i : t \mapsto \gamma(t)$ be a curve on a chart of M, such that $\gamma(0) = x$. A vector X at x tangent to the curve γ is called a tangent vector to M at x. If $\{x^a\}$ are a set of coordinates on U_i, X can be represented by the components

$$X^a = \frac{d}{dt} x^a(\gamma(t)) \bigg|_{t=0} . \tag{195}$$

The collection of the vectors at x tangent to al curves that go through x is called the tangent space $T_x M$ at x.

In a lot of practical situations (and to have an explicit representation) it can be convenient to define a tangent vector by using a function on M. Let f be a function from M to $\mathbb{R}$. One defines a tangent vector by

$$X(f) = \frac{d}{dt} f(\gamma(t)) \bigg|_{t=0} , \tag{196}$$

where now X is represented by a differential operator

$$X = X^a \frac{\partial}{\partial x^a} \equiv X^a \partial_a , \tag{197}$$

so that the set $\{\partial_a\}$ can be considered as a basis for $T_x M$. Note that this definition is consistent with the first one, since if we take the function f to be the coordinate map x^a that maps every point to its a-th coordinate, we get from (196) that $X(x^a) = X^a$. A smooth assignment of a tangent vector at every point $x \in M$ is called a vector field $X(x)$ on M. In subsection 1.4 this is called a section of TM, $X(x) \in \Gamma(M, TM)$.

A.2 Differential forms

Definition 22. The space dual to the tangent space $T_x M$ is the cotangent space $T_x^* M$, that is, $T_x^* M$ is the space of all functionals from $T_x M$ to $\mathbb{R}$. An element of $T_x^* M$ is called a cotangent vector or a 1-form α.

The dual basis to $\{\partial_a\}$ is denoted by $\{dx^a\}$, so that we have that $dx^a(\partial_b) = \delta^a_b$. More generally, this leads to

$$\alpha(X) = \alpha_a dx^a(X^b\partial_b) = \alpha_a X^a. \tag{198}$$

In general a differential form of degree p is an element of the totally anti-symmetric tensor product of p copies of $T^*_x M$. This is accomplished by introducing the wedge product

$$dx^a \wedge dx^b = -dx^b \wedge dx^a. \tag{199}$$

Continuing this process, one can build a basis for the space of all differential forms. The space of p-forms at x is denoted by $\Lambda^p_x M$. Again a smooth assignment of a 1-form at every point of M, is called a 1-form $\alpha(x)$ on M and is a section of T^*M, $\alpha(x) \in \Gamma(M, T^*M) \equiv \Lambda M$. More generally, a p-form on M is an element of $\Lambda^p M$. A general $\alpha_p \in \Lambda^p M$ can be expanded as

$$\alpha_p = \frac{1}{p!}\,\alpha_{a_1 \dots a_p}(x)\, dx^{a_1} \wedge \dots \wedge dx^{a_p}. \tag{200}$$

In this way the product of two differential forms is defined, with the property

$$\alpha_p \wedge \beta_q = (-)^{pq}\beta_q \wedge \alpha_p, \tag{201}$$

as can be seen from the expansion (200). The exterior differential is defined by

$$d = \partial_a\, dx^a \wedge . \tag{202}$$

which is a symbolic notation for

$$d\alpha_p = \frac{1}{p!}\,\partial_a \alpha_{a_1 \dots a_p} dx^a \wedge dx^{a_1} \wedge \dots \wedge dx^{a_p}. \tag{203}$$

This means that d sends p-forms to $(p+1)$-forms, that it is nilpotent, $d^2 = 0$, and that it is an anti-derivation

$$d(\alpha_p \wedge \beta_q) = d\alpha_p \wedge \beta_q + (-)^p \alpha_p \wedge d\beta_q \tag{204}$$

Note the very useful identity $X(f) = df(X)$, as is easily seen by expanding both expressions. A similar expression for a 2-form α is

$$d\alpha(X, Y) = X(\alpha(Y)) - X(\alpha(Y)) - \alpha([X, Y]) \tag{205}$$

where X, $Y \in T_x M$ and $[X, Y] = [X, Y]^a \partial_a$ is the Lie bracket,

$$[X, Y](f) = X(Y(f)) - Y(X(f)). \tag{206}$$

Differential forms are very important when it comes to defining integration on a general manifold. On an n-dimensional manifold M an n-form α_n transforms as a volume element because of the wedge product, so that the integral

$$\int_M \alpha_n \equiv \int_M \alpha(x)\, dx^1 \dots dx^n, \tag{207}$$

is well defined. Here we used that a top form (of maximal dimension) is always characterized by a single function,

$$\alpha_n = \frac{1}{n!}\,\alpha_{a_1 \dots a_n}(x) dx^{a_1} \wedge \dots \wedge dx^{a_n} \Rightarrow \alpha_{a_1 \dots a_n}(x) = \alpha(x)\varepsilon_{a_1 \dots a_n}, \tag{208}$$

and

$$\frac{1}{n!}\,\varepsilon_{a_1\ldots a_n} dx^{a_1}\wedge\ldots\wedge dx^{a_n} = dx^1\wedge\ldots\wedge dx^n \tag{209}$$

Since we always work in Euclidean signature, the totally ant-symmetric tensor is defined by $\varepsilon_{1\ldots n}=1$ and raising indices does not affect the sign. The most important tool we will use, is Stoke's theorem,

$$\int_M d\alpha = \int_{\partial M}\alpha, \tag{210}$$

where ∂M is the boundary of M ($\partial\partial M = 0$).

A form α for which $d\alpha = 0$ is called closed, and if $\alpha = d\beta$ for some form β, α is called exact. A closed form is also called a cocycle and an exact form a coboundary. It is clear that all exact forms are closed (because $d^2 = 0$), but the reverse is not necessarily true on a non-trivial manifold. An important theorem is Poincaré's lemma, which states that locally (on an open set which is contractible) every closed form is exact. The fact that on a general manifold this is not the case globally means that the cohomology defined by closed forms that are not exact contains a lot of information about the topology of a manifold. To be more precise, one defines the group of p-cocycles $Z^p(M,\mathbb{R})$ and the group of p-coboundaries $B^p(M,\mathbb{R})$ as follows

$$Z^p(M,\mathbb{R}) = \{\alpha\in\Lambda^p M|d\alpha=0\}, \tag{211}$$

$$B^p(M,\mathbb{R}) = \{\alpha\in\Lambda^p M|\alpha=d\beta,\,\beta\in\Lambda^{p-1}M\}. \tag{212}$$

The p-th de Rham cohomology class is then defined as the quotient,

$$H^p(M,\mathbb{R}) = Z^p(M,\mathbb{R})/B^p(M,\mathbb{R}), \tag{213}$$

so that a general element of $H^p(M,\mathbb{R})$ is an equivalence class under the equivalence relation,

$$\alpha_p\sim\beta_p \quad \text{iff} \quad \alpha_p = \beta_p + d\gamma_{p-1}, \tag{214}$$

and we denote their common equivalence class by $[\alpha_p]=[\beta_p]\in H^p(M,\mathbb{R})$. A period of an element $[\alpha]\in H^p(M,\mathbb{R})$ over a boundaryless submanifold $C\in M$ is defined by

$$([\alpha],C) = \int_C \alpha, \tag{215}$$

and because of Stoke's theorem this is independent of the choice of representative of the equivalence class. Some cohomology classes, like Chern classes (see section 3), are known to have integral periods, $([c_n(F)],M)\in\mathbb{Z}$. We denote these integral cohomology groups by $H^p(M,\mathbb{Z})$.

Note that if $\dim M = m$, $\Lambda^p M$ and $\Lambda^{m-p}M$ have the same dimension. Given a metric g_{ab} on M, one can define an isomorphism between the two called Hodge duality. When α_p is given by (200), its Hodge dual $*\alpha_{m-p}$ is defined by

$$*\,\alpha_{m-p} = \frac{1}{p!(m-p)!}\,\sqrt{g}\,\alpha_{a_1\ldots a_p}\varepsilon^{a_1\ldots a_p}{}_{a_{p+1}\ldots a_m}\,dx^{a_{p+1}}\wedge\ldots\wedge dx^{a_m}. \tag{216}$$

For Euclidean signature spaces, one finds $** = (-)^{p(m-p)}$. For example, for a 2-form in 4-dimensional Euclidean space, we have $** = 1$. This means that a self-duality condition is well defined.

A.3 Push-forwards and pull-backs

Definition 23. Given a map $f : M \to N$, there is always an induced map $f_* : T_xM \to T_{f(x)}N$ called the push-forward (or differential map) of f. This sends a vector X tangent to a curve γ at $x = \gamma(0) \in M$ to a vector f_*X tangent to the curve $f \circ \gamma$ at the point $y = f(x) = f(\gamma(0)) \in N$.

Concretely, this means that, if $\{y^a\}$ are a set of coordinates for $y \in V \subset N$, where V contains $f(x)$ the components of f_*X are

$$(f_*X)^a = \frac{d}{dt} y^a(f(\gamma(t)))\bigg|_{t=0}. \tag{217}$$

Again, sometimes it is convenient to define f_*X by using an auxiliary function $g : N \to \mathbb{R}$. A definition equivalent to the previous one is

$$f_*X(g) = X(g \circ f). \tag{218}$$

From this definition it is easy to express f_*X in terms of X. First of all, using coordinate bases for the coordinates $\{x^a\}$ on $U \subset M$, where U contains x, and the $\{y^a\}$ defined previously, we find

$$(f_*X)^b \frac{\partial}{\partial y^b} g(y) = X^b \frac{\partial}{\partial x^b}(g(f(x))). \tag{219}$$

Note that on the left hand side, $g(y)$ means that g is expressed in the coordinates $\{y^a\}$, while on the right hand side, $g(f(x))$ means that g is now expressed in the coordinates $\{x^a\}$ by means of the function f. The latter is usually simply denoted by $g(f(x)) \equiv g(x)$. Choosing now $g = y^a$, the a-th coordinate map on $V \subset N$, we find

$$(f_*X)^a = X^b \frac{\partial y^a(x)}{\partial x^b}. \tag{220}$$

We find that the push-forward of X under the map f is simply expressed in terms of the Jacobian of $f = y(x)$. An important property of the push-forward is that for $f : M \to N$ and $h : N \to P$,

$$(h \circ f)_* = h_* f_* \,. \tag{221}$$

Definition 24. Given a map $f : M \to N$, there always exists an induced map $f^* : T^*_{f(x)}N \to T^*_xM$, called the pull-back of f. For an $X \in T_xM$, the pull-back of a 1-form α is given by

$$f^*\alpha(X) = \alpha(f_*X). \tag{222}$$

Using (198) and (220), we find,

$$(f^*\alpha)_b X^b = \alpha_b (f_*X)^b = \alpha_b X^c \frac{\partial y^b(x)}{\partial x^c}. \tag{223}$$

Choosing now, $X = \partial/\partial x^a$, so that $X^b = \delta^b_a$, leads to

$$(f^*\alpha)_a = \alpha_b \frac{\partial y^b(x)}{\partial x^a}. \tag{224}$$

Again, the pull-back of α under the map f is simply expressed in terms of the Jacobian of $f = y(x)$.

One can easily extend this to a definition of the pull-back of a p-form α. For $X_i \in T_x M$,

$$f^*\alpha(X_1, X_2, ..., X_p) = \alpha(f_* X_1, f_* X_2, ..., f_* X_p) \tag{225}$$

Now the induced map is $f^* : \Lambda^p_{f(x)} N \to \Lambda^p_x M$ and in component form we find,

$$f^*\alpha_{a_1 \dots a_p}(x) = \alpha_{b_1 \dots b_p}(y(x)) \frac{\partial y^{b_1}}{\partial x^{a_1}} \dots \frac{\partial y^{b_p}}{\partial x^{a_p}}. \tag{226}$$

The most important properties of the pull-back of a p-form we will use are,

$$d(f^*\alpha) = f^* d\alpha, \tag{227}$$

$$(h \circ f)^* = f^* h^*, \tag{228}$$

$$f^*(\alpha \wedge \beta) = f^*\alpha \wedge f^*\beta, \tag{229}$$

with $f : M \to N$, $h : N \to P$, $\alpha \in \Lambda^p N$ and $\beta \in \Lambda^q N$. These identities are not too difficult to prove using their component form.

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